J. Math. Anal. Appl. 465 (2018) 673–689

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Niebur–Poincaré series and traces of singular moduli

Steffen Löbrich

ARTICLE INFO

Article history:
Received 27 June 2017
Available online 14 May 2018
Submitted by B.C. Berndt

Keywords:
Meromorphic modular forms
Polar harmonic Maass forms
Quadratic forms
Traces of singular moduli
Regularized inner products

ABSTRACT

We compute the Fourier coefficients of analogues of Kohnen and Zagier's modular forms $f_{k,\Delta}$ of weight 2 and negative discriminant. These functions can also be written as twisted traces of certain weight 2 Poincaré series with evaluations of Niebur–Poincaré series as Fourier coefficients. This allows us to study twisted traces of singular moduli in an integral weight setting. In particular, we recover explicit series expressions for twisted traces of singular moduli and extend algebraicity results by Bengoechea to the weight 2 case. We also compute regularized inner products of these functions, which in the higher weight case have been related to evaluations of higher Green's functions at CM-points.

1. Introduction

For a positive discriminant Δ and an integer $k > 1$, Zagier [24] introduced the weight $2k$ cusp forms (in a different normalization)

$$f_{k,\Delta}(\tau) := \frac{\Delta^{k-\frac{1}{2}}}{2\pi} \sum_{Q\in\mathcal{Q}_\Delta} Q(\tau,1)^{-k}, \tag{1.1}$$

where $\mathcal{Q}_\Delta$ denotes the set of binary integral quadratic forms of discriminant Δ. The functions $f_{k,\Delta}$ were extensively studied by Kohnen and Zagier and have several applications. For example, they used these functions to construct the kernel function for the Shimura and Shintani lifts and to prove the non-negativity of twisted central L-values [21]. Furthermore, the even periods

$$\int_0^\infty f_{k,\Delta}(it)t^{2n}dt, \quad (0 \le n \le k-1)$$

of the $f_{k,\Delta}$ are rational [22]. Bengoechea [2] introduced analogous functions for negative discriminants and showed that their Fourier coefficients are algebraic for small k. These functions are no longer holomorphic,

E-mail address: s.loebrich@uva.nl.

https://doi.org/10.1016/j.jmaa.2018.05.034
0022-247X/

but have poles at the CM-points of discriminant Δ. They were realized as regularized theta lifts by Bringmann, Kane, and von Pippich [7] and Zemel [26]. Moreover, Bringmann, Kane, and von Pippich related regularized inner products of the $f_{k,\Delta}$ to evaluations of higher Green's functions at CM-points.

The right-hand side of (1.1) does not converge for $k = 1$. However, one can use Hecke's trick to obtain weight 2 analogues of the $f_{k,\Delta}$. These were introduced by Zagier [24] and further studied by Kohnen [20]. The aim of this paper is to analyze these weight 2 analogues for negative discriminants. Here we deal with generalizations $f^*_{d,D,N}$ for a level N, a discriminant d, and a fundamental discriminant D of opposite sign (see Definition 3.1). The $f^*_{d,D,N}$ transform like modular forms of weight 2 for $\Gamma_0(N)$ and have simple poles at the Heegner points of discriminant dD and level N. Let $\mathcal{Q}_{dD,N}$ denote the set of quadratic forms $[a, b, c]$ of discriminant dD with $a > 0$ and $N|a$, χ_D the generalized genus character associated to D, and $H(d, D, N)$ the twisted Hurwitz class number of discriminants d, D and level N (see Subsection 2.2 for precise definitions). Then we obtain the following Fourier expansion ($v := \mathrm{Im}(\tau)$ throughout).

Theorem 1.1. *For $v > \frac{\sqrt{|dD|}}{2}$, we have*

$$f^*_{d,D,N}(\tau) = -\frac{3H(d,D,N)}{\pi\,[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]\,v} - 2\sum_{n\geq 1}\sum_{\substack{a>0\\ N|a}} S_{d,D}(a,n)\sinh\left(\frac{\pi n\sqrt{|dD|}}{a}\right)e(n\tau),$$

where $e(w) := e^{2\pi i w}$ for all $w \in \mathbb{C}$ and

$$S_{d,D}(a,n) := \sum_{\substack{b \pmod{2a}\\ b^2\equiv dD \pmod{4a}}} \chi_D\left(\left[a, b, \frac{b^2 - dD}{4a}\right]\right) e\left(\frac{nb}{2a}\right).$$

Remark. The exponential sums $S_{d,D}$ also occur for example in [13] and [20].

Note that we obtain a non-holomorphic term in the Fourier expansion of $f^*_{d,D,N}$, just like in the case of the non-holomorphic weight 2 Eisenstein series E_2^* (see Subsection 2.1). Therefore, in contrast to the higher weight case, the $f^*_{d,D,N}$ are in general no longer meromorphic modular forms, but *polar harmonic Maass forms*. This class of functions is defined and studied in Subsection 2.3.

We also use a different approach to compute the coefficients of the $f^*_{d,D,N}$, writing them as traces of certain Poincaré series denoted by $H^*_N(z,\cdot)$ (see Proposition 2.5). The $H^*_N(z,\cdot)$ are weight 2 analogues of Petersson's Poincaré series and were introduced by Bringmann and Kane [5] to obtain an explicit version of the Riemann–Roch Theorem in weight 0. We obtain the following different Fourier expansion of the $f^*_{d,D,N}$, realizing their coefficients as twisted traces of the Niebur–Poincaré series $j_{N,n}$ (see Definition 2.1 and Theorem 2.4).

Theorem 1.2. *For $v > \max\left\{\frac{\sqrt{|dD|}}{2}, 1\right\}$, we have*

$$f^*_{d,D,N}(\tau) = -\frac{3H(d,D,N)}{\pi\,[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]\,v} - \sum_{n>0} \mathrm{tr}_{d,D,N}\left(j_{N,n}\right)e(n\tau).$$

An interesting phenomenon occurs when $\Gamma_0(N)$ has genus 0. Subgroups of this type and their Hauptmoduln play a fundamental role in Monstrous Moonshine (see for example [11] for a classical and [14] for a more modern treatment). When we apply the suitably normalized n-th Hecke operator T_n to the Hauptmodul J_N for $\Gamma_0(N)$, then the Niebur–Poincaré series $j_{N,n}$ coincides with T_nJ_N, up to an additive constant. Zagier [25] showed that, for discriminants $d < 0$ and $D > 0$, the functions