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A free-boundary problem for Euler flows with constant vorticity on the sphere

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ABSTRACT

A free-boundary problem for two-dimensional Euler flows with uniform vorticity on the surface of sphere is considered using the stereographic projection and the argument principle in complex variables. With the constant speed condition on the boundary, a circle turns out to be the unique solution on the sphere.

1. Introduction

The fluid mechanics on the surface of sphere has been an important subject and recently attracting a lot of interest due to various geophysical phenomena of atmosphere and ocean. If one focuses on the dynamics of vortices among many specific topics, there appears again extensive literature published for decades which cannot be summarized briefly.

In particular, various aspects of the dynamics of point vortices have been considered for an inviscid incompressible flow on the surface of the sphere. It is then found that some of important properties of point vortices are transferred from the planar case to the spherical case with no modification due to the highly symmetric geometry of the sphere. For example, the integrability condition and collapse phenomenon of point vortices are almost same in two cases. (See [5] for details.)

Compared with this, there have been relatively few results on the dynamics of vortex patch of uniform (nonzero) vorticity in a region, which might be regarded as a more realistic model than point vortex for vortex flow [6]. Therefore, it is necessary to investigate further the dynamics of vortex patch on the sphere [1].

In addition, the vortex patch is closely related to a nearly inviscid flow or Prandtl–Batchelor flow characterizing a possible inviscid limit flow with closed nested streamlines and uniform vorticity region in the plane [3]. The corresponding theory for the sphere is also discussed briefly in [8].

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Under this background, we study a free-boundary problem determining the shape of constant vortex patch on the sphere under the constant speed condition on the (unknown) boundary. The planar case was previously studied by the present author [4] to confirm the uniqueness of the circular boundary (and the rigid body rotation). By the technique of stereographic projection and argument principle in complex variable formulation, we here establish the same conclusion for the spherical case.

2. A free-boundary problem on the sphere

We are interested in the following free-boundary problem for an inviscid incompressible fluid flow on the sphere:

Let S^2 be the surface of a two dimensional sphere of unit radius. Determine a simply connected smooth domain $D_0 \subset S^2$ throughout which the vorticity is constant $\omega_0 > 0$ and also a constant speed $q_0 > 0$ is given along the unknown boundary ∂D_0.

In mathematical expression, this corresponds to

$$\nabla^2_{S^2}\psi = -\omega_0 \quad \text{on } D_0 \tag{1}$$

$$\psi = 0 \quad \text{on } \partial D_0 \tag{2}$$

$$\frac{\partial \psi}{\partial n} = q_0 \quad \text{on } \partial D_0 \tag{3}$$

for an unknown (stream) function $\psi = \psi(\theta, \phi)$ where $0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$ represent the latitude and the longitude, respectively. (Here, $\nabla^2_{S^2}$ is the Laplace–Beltrami operator

$$\nabla^2_{S^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

on the sphere and n is directed inwardly normal to the boundary.)

Briefly speaking this problem corresponds to determine the shape of a vortex patch for a two-dimensional incompressible Euler flow moving with constant vorticity ω_0 and constant boundary speed q_0. At once, we know that this problem is subject to the compatibility constraint

$$\omega_0 \cdot \text{Surface Area}(D_0) = q_0 \cdot \text{Length}(\partial D_0) \tag{4}$$

from Stokes' integral identity. This produces a necessary relation among the area of D_0, the length of ∂D_0 and the speed q_0 on ∂D_0. From now, we assume this condition holds.

For the planar case, the analogous problem is studied by the author [4] by complex variable approach, which concluded that D_0 is a circle with radius $r = \frac{2q_0}{\omega_0}$. The corresponding motion of fluid, then is a rigid-body rotation as reconfirmed in [7].

One characteristic of current problem is to find the whole boundary of the domain. Typical free-boundary problems in fluid dynamics ask to resolve only a part of the (whole) boundary with some known parts of the boundary, e.g. fixed two end points and (or) some portion of boundary (see Friedman [2]). This peculiar property results in a difficulty to construct a corresponding suitable variational formulation. In the two-dimensional setting, however, we overcome this difficulty by a complex variable approach.

Specifically we adopt the stereographic projection and consider the problem in the complex plane instead of the sphere [1]. So let us introduce the complex variable $z = x + iy$, and project the free-boundary domain D_0 onto D in the z-plane. Here the relation is given by

$$z = re^{i\phi}, \quad r = \cot\left(\frac{\theta}{2}\right),$$

where (r, ϕ) correspond to polar coordinates of z-plane. See Fig. 1.