J. Math. Anal. Appl. 464 (2018) 1158–1166

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Preduals for spaces of operators involving Hilbert spaces and trace-class operators ☆

Hannes Thiel

Mathematisches Institut, Universität Münster, Einsteinstr. 62, 48149 Münster, Germany

ARTICLE INFO

Article history:
Received 27 March 2018
Available online 23 April 2018
Submitted by G. Corach

Keywords:
Preduals
Unique predual
Banach algebras
Complemented subspace

ABSTRACT

Continuing the study of preduals of spaces $\mathcal{L}(H, Y)$ of bounded, linear maps, we consider the situation that H is a Hilbert space. We establish a natural correspondence between isometric preduals of $\mathcal{L}(H, Y)$ and isometric preduals of Y. The main ingredient is a Tomiyama-type result which shows that every contractive projection that complements $\mathcal{L}(H, Y)$ in its bidual is automatically a right $\mathcal{L}(H)$-module map. As an application, we show that isometric preduals of $\mathcal{L}(\mathcal{S}_1)$, the algebra of operators on the space of trace-class operators, correspond to isometric preduals of $\mathcal{S}_1$ itself (and there is an abundance of them). On the other hand, the compact operators are the unique predual of $\mathcal{S}_1$ making its multiplication separately weak* continuous.

1. Introduction

An isometric predual of a Banach space X is a Banach space F together with an isometric isomorphism $X \cong F^*$. Every predual induces a weak* topology. Due to the importance of weak* topologies, it is interesting to study the existence and uniqueness of preduals; see the survey [5] and the references therein.

Given Banach spaces X and Y, let $\mathcal{L}(X, Y)$ denote the space of operators from X to Y. Every isometric predual of Y induces an isometric predual of $\mathcal{L}(X, Y)$: If $Y \cong F^*$, then $\mathcal{L}(X, Y) \cong (X \hat{\otimes} F)^*$.

Problem 1.1. Find conditions on X and Y guaranteeing that every isometric predual of $\mathcal{L}(X, Y)$ is induced from an isometric predual of Y.

Given reflexive spaces X and Y, Godefroy and Saphar show that $X \widehat{\otimes} Y^*$ is the strongly unique isometric predual of $\mathcal{L}(X, Y)$; see [6, Proposition 5.10]. In particular, in this case every isometric predual $\mathcal{L}(X, Y)$ is induced from Y.

☆ The author was partially supported by the Deutsche Forschungsgemeinschaft (SFB 878 *Groups, Geometry & Actions*).
E-mail address: hannes.thiel@uni-muenster.de.
URL: http://www.math.uni-muenster.de/u/hannes.thiel/.

https://doi.org/10.1016/j.jmaa.2018.04.051

The main result of this paper extends this to the case that X is a Hilbert space H and Y is arbitrary: Every isometric predual of $\mathcal{L}(H,Y)$ is induced from an isometric predual of Y; see Theorem 2.7. In particular, $\mathcal{L}(H,Y)$ has a (strongly unique) isometric predual if and only if Y does; see Corollary 2.8.

To obtain these results, we use that isometric preduals of Y naturally correspond to contractive projections $\mathcal{L}(X,Y)^{**} \to \mathcal{L}(X,Y)$ that are right $\mathcal{L}(X)$-module maps and have weak* closed kernel; see [4, Theorem 5.7]. Hence, we are faced with:

Problem 1.2. Find conditions on X and Y guaranteeing that every contractive projection $\mathcal{L}(X,Y)^{**} \to \mathcal{L}(X,Y)$ is automatically a right $\mathcal{L}(X)$-module map.

It was shown by Tomiyama that every contractive projection from a C^*-algebra A onto a sub-C^*-algebra B is automatically a B-bimodule map. We therefore consider a positive solution to Problem 1.2 a Tomiyama-type result. Adapting the proof of Tomiyama's result, we obtain a positive solution to Problem 1.2 whenever X is a Hilbert space; see Theorem 2.4.

In Section 3, we show that our results also hold when the Hilbert space H is replaced by the space of trace-class operators $\mathcal{S}_1(H)$. It follows that isometric preduals of $\mathcal{L}(\mathcal{S}_1(H))$ naturally correspond to isometric preduals of $\mathcal{S}_1(H)$; see Example 3.8. We note that $\mathcal{S}_1(H)$ – and consequently $\mathcal{L}(\mathcal{S}_1(H))$ – has many different isometric preduals. On the other hand, we show that the 'standard' predual of compact operators is the unique predual of $\mathcal{S}_1(H)$ making its multiplication separately weak* continuous; see Theorem 3.9.

Acknowledgments. The author would like to thank Eusebio Gardella and Tim de Laat for valuable feedback.

Notation. Given Banach spaces X and Y, an *operator* from X to Y is a bounded, linear map $X \to Y$. The space of such operators is denoted by $\mathcal{L}(X,Y)$. The projective tensor product of X and Y is denoted by $X\hat{\otimes}Y$. We identify X with a subspace of its bidual, and we let $\kappa_X \colon X \to X^{**}$ denote the inclusion. A *projection* $\pi\colon X^{**} \to X$ is an operator satisfying $\pi(x) = x$ for all $x \in X$.

2. Preduals involving Hilbert spaces

Throughout, X and Y denote Banach spaces. For conceptual reasons, it is useful to consider preduals of X as subsets of X^*. More precisely, a closed subspace $F \subseteq X^*$ is an (isometric) predual of X if for the inclusion map $\iota_F \colon F \to X^*$, the transpose map $\iota_F^* \colon X^{**} \to F^*$ restricts to an (isometric) isomorphism $X \to F^*$.

The space X is said to have a *strongly unique isometric predual* if there exists an isometric predual $F \subseteq X^*$ and if $F = G$ for every isometric predual $G \subseteq X^*$. Every reflexive space X has a strongly unique isometric predual, namely X^*.

2.1. The space $\mathcal{L}(X,Y)$ has a natural $\mathcal{L}(Y)$-$\mathcal{L}(X)$-bimodule structure. Given $a \in \mathcal{L}(X)$, the action of a is given by $R_a \colon \mathcal{L}(X,Y) \to \mathcal{L}(X,Y)$, $R_a(f) := f \circ a$, for $f \in \mathcal{L}(X,Y)$. Thus, a acts by precomposing on the right of $\mathcal{L}(X,Y)$. Similarly, the action of $b \in \mathcal{L}(Y)$ is given by postcomposing on the left, that is, by $L_b \colon \mathcal{L}(X,Y) \to \mathcal{L}(X,Y)$, $L_b(f) := b \circ f$, for $f \in \mathcal{L}(X,Y)$.

We obtain a $\mathcal{L}(X)$-$\mathcal{L}(Y)$-bimodule structure on $\mathcal{L}(X,Y)^*$. The *left* action of $a \in \mathcal{L}(X)$ on $\mathcal{L}(X,Y)^*$ is given by R_a^*. The *right* action of $b \in \mathcal{L}(Y)$ on $\mathcal{L}(X,Y)^*$ is given by L_b^*. Similarly, we obtain a $\mathcal{L}(Y)$-$\mathcal{L}(X)$-bimodule structure on $\mathcal{L}(X,Y)^{**}$.

Given a C^*-algebra A and $a, b, x, y \in A$ with $a^*b = 0$, we have $\|ax + by\|^2 \leq \|ax\|^2 + \|by\|^2$, which is an analog of Bessel's inequality; see [1, II.3.1.12, p.66]. We first prove two versions of this result in a more general context.