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Early-warning signals for bifurcations in random dynamical systems with bounded noise

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ABSTRACT

We consider discrete-time one-dimensional random dynamical systems with bounded noise, which generate an associated set-valued dynamical system. We provide necessary and sufficient conditions for a discontinuous bifurcation of a minimal invariant set of the set-valued dynamical system in terms of the derivatives of the so-called extremal maps. We propose an algorithm for reconstructing the derivatives of the extremal maps from a time series that is generated by iterations of the original random dynamical system. We demonstrate that the derivative reconstructed for different parameters can be used as an early-warning signal to detect an upcoming bifurcation, and apply the algorithm to the bifurcation analysis of the stochastic return map of the Koper model, which is a three-dimensional multiple time scale ordinary differential equation used as prototypical model for the formation of mixed-mode oscillation patterns. We apply our algorithm to data generated by this map to detect an upcoming transition.

1. Introduction

There has been a steadily increasing interest into dealing with sudden changes in the behaviour of dynamical systems, sometimes referred to as *critical transitions* and *tipping points* in the applied sciences. Many critical transitions can be modelled mathematically using differential equations involving *bifurcations*. For instance, medical conditions such as asthma attacks and epileptic seizures can change quickly from regular to irregular behaviour, the financial markets are known to suddenly break trends in a crisis, and climate conditions and ecological environments can change rather abruptly.

The understanding of the dynamical behaviour near critical transitions is very important, since it enables human interaction to attenuate or control the consequences of critical transitions, but predicting critical

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transitions before they are reached is notably hard. Recent results by scientists working in different fields of applications suggest the existence of generic early-warning signals (such as autocorrelation and variance) that could indicate for a large class of systems if a critical transition is being approached [16,11].

In low-dimensional autonomous (time-independent) dynamical systems, critical transitions have been extensively studied in the context of bifurcation theory, by explaining and classifying ways in which attractors can lose stability and give rise to new types of behaviours, from simple transitions between stationarity and oscillations, to transitions between predictable and intrinsically unpredictable (chaotic) dynamics [4,13]. There is a central assumption of very slow (adiabatic) variation of parameters in the classical bifurcation theory, and the problem of critical transitions in complex systems and their early-warning signals motivates the need to deal with transitions in nonautonomous (time-dependent) and random dynamical systems [7,1]. Critical transitions in form of rate-induced tipping in nonautonomous dynamical systems have recently been studied in [2,15]

In this paper, we consider discrete-time random dynamical systems with bounded noise, which give rise to a set-valued dynamical system. We study bifurcations of the set-valued dynamical system in form of discontinuous changes in the minimal invariant sets (which correspond to critical transitions), and we address the question of an early warning to predict such transitions. We provide an algorithm that approximates the derivatives of the extremal mappings of one-dimensional set-valued dynamical systems from a time series generated by the corresponding random dynamical system with bounded noise. The use of derivatives of extremal maps as early-warning signals has not been established before. We show that the derivative is equal to 1 at a bifurcation point necessarily, and provides a universal threshold for this reason, which does not depend on the type of system or the type of bounded noise.

We apply our theoretical results to the return map in the Koper model [8]. This model is a prototypical example exhibiting mixed-mode oscillations (MMOs) [5]. It is a system of ordinary differential equations with one fast and two slow variables. Formally, a generic return map to a codimension-one section would be a two-dimensional map but the strong contraction in one direction implies that the system has an effectively one-dimensional return map [6]. We study a bounded noise perturbation of the Koper model leading to a random return map. In particular, we are interested in the detection of bifurcations that induce a change in the MMO pattern. We demonstrate using numerical simulation and analysis of the resulting time series, that the derivatives of the extremal maps can be used as an early-warning sign for the changing patterns.

This paper is organized as follows. In Section 2, we provide necessary and sufficient conditions for bifurcations of minimal invariant sets. We formulate the conditions in terms of the derivative of extremal maps, which define the shape of the graph of the set-valued map. In Section 3, we present an algorithm for estimating the derivatives of the extremal maps from a time series generated by the corresponding random dynamical system. We prove also a probability statement about the reliability of the estimate containing an explicit dependence on the sample size. In Section 4, we apply our method to predict changes in MMOs in the Koper model.

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2. Bifurcations of set-valued mappings

Consider the dynamics of continuous mappings $f : \mathbb{R} \to \mathbb{R}$ perturbed by additive bounded noise, given by the random difference equation