J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Complex symmetric weighted composition operators on $\mathcal{H}_\gamma(\mathbb{D})$ ☆

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A R T I C L E I N F O

Article history:
Received 12 February 2018
Available online xxxx
Submitted by T. Ransford

Keywords:
Hardy space
Bergman space
Weighted composition operator
Conjugation
Complex symmetry

A B S T R A C T

We study the complex symmetric structure of weighted composition operators of the form $W_{\psi,\varphi}$ on the Hilbert space $\mathcal{H}_\gamma(\mathbb{D})$ of holomorphic functions over the open unit disk $\mathbb{D}$ with reproducing kernels $K_w^{(\gamma)} = (1-\overline{w}z)^{-\gamma}$, where $\gamma \in \mathbb{N}$. First, we consider conjugations on $\mathcal{H}_\gamma(\mathbb{D})$ of the form $\mathcal{A}_{u,v}f = u \cdot \overline{f \circ \overline{v}}$ (such conjugations are also known as weighted composition conjugations) and characterize them into two classes, denoted by $\mathcal{C}_1$ and $\mathcal{C}_2$. Then, we obtain explicit conditions for $W_{\psi,\varphi}$ when it is $\mathcal{C}_1$-symmetric and $\mathcal{C}_2$-symmetric respectively.

1. Introduction

The general study of complex symmetry has origins that can be traced back to both operator theory and complex analysis, and was initiated by Garcia and Putinar in [4,5]. Thereafter, a number of papers were devoted to the topic and contributed significantly to the study of complex symmetry (see [3] and the references therein). In the past decade, complex symmetric operators have become particularly important in both theoretical aspects and applications. Renewed interest in non-Hermitian quantum mechanics and spectral analysis of certain complex symmetric operators has brought forth more research into complex symmetry. Present results show that bounded complex symmetric operators are very diverse, which include the likes of Volterra integration operator, normal operator, compressed Toeplitz operators, etc. Recently, in [11], some interpolation properties of complex symmetric operators on Hilbert spaces, with application to complex symmetric Toeplitz operators, have been studied.

☆ Supported in part by MOE's AcRF Tier 1 grant M4011724.110 (RG128/16).

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https://doi.org/10.1016/j.jmaa.2018.03.071
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1.1. Basic definitions and notation

Let $\mathcal{H}$ denote a separable complex Hilbert space. A mapping $\mathcal{A}$ acting on $\mathcal{H}$ is said to be *anti-linear* (or *conjugate-linear*) if for all $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{C}$,

$$\mathcal{A}(\alpha x + y) = \bar{\alpha}\mathcal{A}x + \mathcal{A}y.$$

The *boundedness* of an anti-linear operator is defined in the same manner as a linear operator.

For a bounded anti-linear operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}$, its *adjoint* $\mathcal{A}^*$ satisfies

$$\langle \mathcal{A}x, y\rangle = \overline{\langle x, \mathcal{A}^*y\rangle} = \langle \mathcal{A}^*y, x\rangle$$

for all $x, y \in \mathcal{H}$. We say that $\mathcal{A}$ is *unitary* if $\mathcal{A}\mathcal{A}^* = \mathcal{A}^*\mathcal{A} = I$, where $I : \mathcal{H} \to \mathcal{H}$ is the identity operator. Also, $\mathcal{A}$ is *self-adjoint* if $\mathcal{A} = \mathcal{A}^*$, i.e. for all $x, y \in \mathcal{H}$, $\langle \mathcal{A}x, y\rangle = \langle \mathcal{A}y, x\rangle$.

An anti-linear operator $\mathcal{C} : \mathcal{H} \to \mathcal{H}$ is a **conjugation** if $\mathcal{C}$ is **(a)** *involutive*, i.e. $\mathcal{C}^2 = I$ and **(b)** *isometric*, i.e. $\|\mathcal{C}x\| = \|x\|$ for all $x \in \mathcal{H}$. Note that a conjugation is necessarily bounded as it is isometric. A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is said to be **complex symmetric** if there exists a conjugation $\mathcal{C} : \mathcal{H} \to \mathcal{H}$ such that $T = \mathcal{C}T^*\mathcal{C}$ (equivalent to either $T\mathcal{C} = \mathcal{C}T^*$ or $T^* = \mathcal{C}T\mathcal{C}$). We also say that T is a $\mathcal{C}$-symmetric operator.

A *reproducing kernel Hilbert space* (RKHS) on a set X is a Hilbert space on X for which the linear evaluation functional on X is bounded. By a simple application of the Riesz representation theorem, there is a unique vector in the RKHS, say K_w, such that for all f in the RKHS,

$$f(w) = \langle f, K_w\rangle$$

for all $w \in X$. We call K_w the *reproducing kernel*. It is well known that a reproducing kernel uniquely defines a RKHS. We refer the interested reader to [12] for more details on RKHS.

Let $\mathbb{D}$ be an open unit disk on the complex plane. For all $w, z \in \mathbb{D}$ and $\gamma \in \mathbb{N}$, consider the reproducing kernel

$$K_w^{(\gamma)}(z) = (1 - \overline{w}z)^{-\gamma}. \tag{1.1}$$

We denote by $\mathcal{H}_\gamma(\mathbb{D})$, the Hilbert space of holomorphic functions on $\mathbb{D}$ with reproducing kernel $K_w^{(\gamma)}(z)$. It is worth noting that when $\gamma = 1$, we get the classical Hardy space, while when $\gamma = 2$, we obtain the classical Bergman space. In fact, such a space was studied in [10] for several complex variables and $\gamma > 0$. However, we will restrict our study to just $\gamma \in \mathbb{N}$. This restriction will be justified later on.

For any $w, z \in \mathbb{D}$, the norm of $K_w^{(\gamma)}(z)$ is given by

$$\left\|K_w^{(\gamma)}\right\| = \left(1 - |w|^2\right)^{-\frac{\gamma}{2}}.$$

Hence we can write the normalized reproducing kernel as

$$k_w^{(\gamma)}(z) = \frac{K_w^{(\gamma)}(z)}{\left\|K_w^{(\gamma)}\right\|} = \frac{\left(1 - |w|^2\right)^{\frac{\gamma}{2}}}{(1 - \overline{w}z)^\gamma}. \tag{1.2}$$

Now, let $\psi : \mathbb{D} \to \mathbb{C}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic functions. For all $f \in \mathcal{H}_\gamma(\mathbb{D})$ and $z \in \mathbb{D}$, the functions ψ and φ induce a linear *weighted composition operator* $W_{\psi,\varphi}$ on $\mathcal{H}_\gamma(\mathbb{D})$ defined by

$$W_{\psi,\varphi}f(z) = \psi(z) \cdot f(\varphi(z)).$$