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Compact and locally dense leaves of a closed one-form foliation

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A B S T R A C T

We study a foliation defined by a possibly singular smooth closed one-form on a connected smooth closed orientable manifold. We prove two bounds on the total number of homologically independent compact leaves and of connected components of the union of all locally dense leaves, which we call minimal components. In particular, we generalize the notion of minimal components, previously used in the context of Morse form foliations, to general foliations. Finally, we give a condition for the form foliation to have only closed leaves (closed in the complement of the singular set).

1. Introduction

Let ω be a smooth closed one-form on a connected smooth closed orientable manifold M and $\operatorname{Sing}\omega$ the set of its singularities. This form defines on $M \setminus \operatorname{Sing}\omega$ a codimension-one foliation $\mathcal{F}_\omega$. This type of foliations is important in applications to physics, e.g., in supergravity theory [2,3].

Smooth closed one-forms define an important class of foliations: foliations without holonomy; moreover, any codimension-one foliation without holonomy is topologically equivalent to a foliation defined by a smooth closed one-form [25]. A subset of smooth closed one-forms, Morse forms (locally the differential of a Morse function), is well-studied.

A leaf of a codimension-one foliation is either *proper* (locally closed, hence it is a regular submanifold), *locally dense* (its closure has non-empty interior), or *exceptional* (its closure is known to be transversally homeomorphic to a Cantor set). In particular, a compact leaf is proper.

The number of different leaves of each kind, usually up to some equivalence relation, or objects related with such equivalence classes, is an important topological invariant of a foliation. In this paper, we will study the number of locally dense and compact leaves of a closed one-form foliation up to some topological and homological equivalence, respectively.

One construction used (not in this paper) to count equivalence classes of leaves of a foliation (defined on the whole arbitrary manifold M) is *minimal sets*: minimal non-empty closed saturated (i.e., consisting of whole leaves) subsets of M. The number of exceptional minimal sets of a codimension-one foliation on a

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compact manifold is finite; moreover, for arbitrary $k \in \mathbb{N}$ there exists a foliation on a compact manifold with k exceptional minimal sets [17, Theorem 4.1.3]. This number is finite even for a C^0-foliation on a closed manifold [4, Lemma 2.13].

However, each compact leaf is a minimal set in itself, the union of all compact leaves of a closed one-form foliation being open, so if a foliation has a compact leaf, then the number of its minimal sets is infinite. It is even worse for locally dense leaves: a minimal set contains a locally dense leaf L only if $\overline{L} = M$. So minimal sets are not a suitable tool for studying compact and locally dense leaves.

Another construction is connected components of the union of leaves of each type. For a Morse form foliation $\mathcal{F}_\omega$, connected components of the union of locally dense leaves are called *minimal components* [1] and connected components of the union of compact leaves are called *maximal components* [8], or in the case of 2-surfaces, *periodic components* [27].

A Morse form foliation contains only closed (in $M \setminus \operatorname{Sing}\omega$) and locally dense leaves [1,18]. The number $m(\omega)$ of its minimal components, the number $M(\omega)$ of its maximal components, and the number of its singularities are finite [8,9]; moreover (Corollary 2.2):

$$2m(\omega) + M(\omega) \le b_1(M) + |\operatorname{Sing}\omega| - 1,$$
$$m(\omega) + M(\omega) \le b_1'(M) + |\operatorname{Sing}\omega| - 1,$$

where $b_1(M)$ is the Betti number and $b_1'(M)$ is the co-rank of the fundamental group $\pi_1(M)$; the second inequality is exact for each M. The number of closed (in $M \setminus \operatorname{Sing}\omega$) but non-compact leaves of a Morse form foliation is also finite. However, whereas we will show below that $m(\omega)$ can be generalized to arbitrary smooth closed one-form, these inequalities cannot be generalized because both $\operatorname{Sing}\omega$ and $M(\omega)$ can generally be infinite, as in Example 3.1 below.

For the study of compact leaves, homology theory provides more useful tools: obviously, for a form on a compact manifold, the number $c(\omega)$ of homologically independent compact leaves of $\mathcal{F}_\omega$ is finite: $c(\omega) \le b_1(M)$, the first Betti number. For a Morse form foliation, the number $c(\omega)$ of homologically independent compact leaves and the number $m(\omega)$ of minimal components are related:

$$2m(\omega) + c(\omega) \le b_1(M), \qquad \text{[8, Theorem 3.1]}$$
$$m(\omega) + c(\omega) \le b_1'(M). \qquad \text{[9, Theorem 3]}$$

In this paper, we generalize the notion of a minimal component to arbitrary foliations and show that these two bounds hold for arbitrary smooth closed one-forms.

The paper is organized as follows. In Section 2, we give necessary definitions and known facts concerning smooth closed one-form foliations, Morse form foliations, the co-rank $b_1'(M)$ of the fundamental group $\pi_1(M)$, close cohomologous one-forms, and $\mathcal{F}$-saturated sets. In Section 3, we generalize the notion of a minimal component from Morse form foliations to arbitrary smooth closed one-form foliations and study its properties. In Section 4, we introduce extended minimal components, which are open sets in one-to-one correspondence with minimal components. In Section 5, we prove our main theorem: the bounds on $c(\omega)$ and $m(\omega)$ in terms of $b_1'(M)$ and the first Betti number $b_1(M)$. Finally, in Section 6, we apply this theorem to obtain a condition for compactifiability of a smooth closed one-form foliation in terms of $b_1'(M)$.

2. Definitions and useful facts

Unless stated otherwise, we will consider a connected smooth closed orientable n-dimensional manifold M.