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A Riemann solver at a junction compatible with a homogenization limit

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A B S T R A C T

We consider a junction regulated by a traffic lights, with n incoming roads and only one outgoing road. On each road the Phase Transition traffic model, proposed in [6], describes the evolution of car traffic. Such model is an extension of the classic Lighthill–Whitham–Richards one, obtained by assuming that different drivers may have different maximal speed.

By sending to infinity the number of cycles of the traffic lights, we obtain a justification of the Riemann solver introduced in [9] and in particular of the rule for determining the maximal speed in the outgoing road.

1. Introduction

This paper deals with the Phase Transition traffic model, proposed by Colombo, Marcellini, and Rascle in [6], at a junction with $n \geq 2$ incoming roads and only one outgoing road. The aim is to give a mathematical derivation for the solution of the Riemann problem at the crossroad proposed in [9]. We obtain the justification for such solution by using a homogenization procedure.

The traffic model considered in this paper is a system of 2×2 conservation laws; it belongs to the class of macroscopic second order models as the famous Aw–Rascle–Zhang model, see [1,19]. As the name Phase Transition suggests, the model is characterized by two different phases, the free one and the congested one; see [2,4,5,7,8,10,14,16,17] and the references therein for similar descriptions. The Phase Transition model we consider here is derived from the famous Lighthill–Whitham–Richards one [15,18] by assuming that different drivers may have different maximal speeds.

The extension of the Phase Transition model to the case of networks is considered in [9]. The key point for extending a model to a network consists in providing a concept of solution at nodes. A possible way to do this is to construct a Riemann solver at nodes, i.e. a function which associates to each Riemann problem

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at the node a solution. A reasonable Riemann solver has to satisfy the mass conservation, a consistency condition; it should produce waves with negative speed in the incoming edges and with positive speed in the outgoing ones. A Riemann solver satisfying such properties is proposed in [9]. In particular, it prescribes that the maximal speed in the outgoing road is a convex combination of the maximal speed in the incoming arcs. Similar conditions are also present in [11–13].

In this paper we are going to investigate the delicate issue of how the maximal speed *changes* through the junction. To this aim, we consider a single junction regulated by a time-periodic traffic lights. At each time the green light applies only at one incoming road. Vehicles, in the remaining incoming roads, are then stopped, waiting for their green light. With a limit-average procedure, we are able to find the relation between the incoming maximal speeds and the outgoing one. In this way, the maximal outgoing speed turns out to be a convex combination of the n incoming ones and it satisfies the corresponding condition prescribed by the Riemann solver in [9].

The paper is organized as follow. In the next section we recall the 2-Phases Traffic Model introduced in [6] and the solution to the classical Riemann problem along a single road of infinite length. In Section 3 we consider a time periodic traffic lights regulating the intersection and we study the solution in the outgoing road as the time period of the traffic lights tends to 0. More precisely, in Subsection 3.1 we describe in details the solution in the simple situation with $n = 2$ incoming roads and, finally, in Subsection 3.2 we generalize the previous study to the case of $n \geq 2$ incoming roads and we state and prove the main result, concerning the rule for the maximal speed in the outgoing road, by using a limit-average procedure.

2. Notations and the Riemann problem on a single road

The Phase Transition model, introduced in [6], is given by:

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho,\eta)\right) = 0 \\ \partial_t \eta + \partial_x \left(\eta\, v(\rho,\eta)\right) = 0 \end{cases} \qquad \text{with} \qquad v(\rho,\eta) = \min\left\{V_{\max}, \frac{\eta}{\rho}\,\psi(\rho)\right\}, \tag{2.1}$$

where t denotes the time, x the space, $\rho \in [0, R]$ is the traffic density, η is a generalized momentum, $v \in [0, V_{\max}]$ is the speed of cars, and $V_{\max}$ is a uniform bound of the cars' speed.

It is obtained as an extension of the Lighthill–Whitham–Richards model [15,18], by assuming that different drivers have different maximal speed, denoted by the quantity $w = \eta/\rho \in [\check{w}, \hat{w}]$. It is characterized by two phases, the free one and congested one, which are described by the sets

$$F = \left\{(\rho,\eta) \in [0,R] \times [0,\hat{w}R] \colon \check{w}\rho \leq \eta \leq \hat{w}\rho,\ v(\rho,\eta) = V_{\max}\right\}, \tag{2.2}$$

$$C = \left\{(\rho,\eta) \in [0,R] \times [0,\hat{w}R] \colon \check{w}\rho \leq \eta \leq \hat{w}\rho,\ v(\rho,\eta) = \frac{\eta}{\rho}\,\psi(\rho)\right\}, \tag{2.3}$$

see Fig. 1. As in [6,9], we assume the following hypotheses.

(H-1) $R, \check{w}, \hat{w}, V_{\max}$ are positive constants, with $V_{\max} < \check{w} < \hat{w}$.

(H-2) $\psi \in \mathbf{C}^2\left([0,R];[0,1]\right)$ is such that $\psi(0) = 1$, $\psi(R) = 0$, and, for every $\rho \in (0,R)$, $\psi'(\rho) \leq 0$, $\frac{d^2}{d\rho^2}\left(\rho\,\psi(\rho)\right) \leq 0$.

(H-3) Waves of the first family in the congested phase C have negative speed.