Doctopic: Applied Mathematics ARTICLE IN PRESS YJMAA:22159

J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The weak Galerkin finite element method for incompressible flow ☆

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A R T I C L E I N F O

Article history:
Received 16 August 2017
Available online xxxx
Submitted by R.G. Durán

Keywords:
Stable weak Galerkin method
Navier–Stokes equation
Weak embedding inequality
Stability and error analysis

A B S T R A C T

We study the weak Galerkin finite element method for stationary Navier–Stokes problem. We propose a weak finite element velocity-pressure space pair that satisfies the discrete inf-sup condition. This space pair is then employed to construct a stable weak Galerkin finite element scheme without adding any stabilizing term or penalty term. We prove a discrete embedding inequality on the weak finite element space which, together with the discrete inf-sup condition, enables us to establish the unique existence and stability estimates of the discrete velocity and pressure. Then, we derive the optimal error estimates for velocity and pressure approximations in the H^1-norm and L_2-norm, respectively. Numerical experiments are provided to illustrate the theoretical analysis.

1. Introduction

In recent years, weak Galerkin (WG) finite element methods have been developed and analyzed for solving various partial differential equations [2,3,6–16,18,19]. In general, a WG finite element method can be considered as an extension of the standard finite element method where classical derivatives are replaced in the variational equation by the weakly defined derivatives on discontinuous weak functions. There are two main features in WG methods: (1) the weak derivatives are introduced as distributions for weak functions; (2) the weak finite element function $u_h = \{u_h^0, u_h^b\}$ is used in which u_h^0 is totally discontinuous on the partition and the component u_h^b of u_h on element boundary may be independent of the component u_h^0 of u_h in the interior of element.

Although many research works have been done on WG methods, to authors' best knowledge, these works are presented for linear problems and there are no WG finite element works for nonlinear problems in the existing literatures yet. As a class of emerging finite element methods, it is interesting to see how WG

☆ This work was supported by the National Natural Science Funds of China, No. 11371081.
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https://doi.org/10.1016/j.jmaa.2018.04.005
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finite element methods can be extended to nonlinear partial differential equations, and this motivates us to develop and analyze a WG finite element method for the stationary Navier–Stokes problem.

At present, some stabilized WG finite element methods for Stokes problems [2,10,15,16] have appeared in the literature, but these methods are not applicable to Navier–Stokes problem because of the nonlinear convection term. In conventional finite element methods for Stokes and Navier–Stokes equations, a standard requirement for the velocity-pressure space pair is the so-called *inf-sup* condition. The importance of ensuring the inf-sup condition is well understood. Numerical experiments show that the violation of the inf-sup condition often leads to nonphysical oscillations of the discrete solutions. In order to circumvent the inf-sup condition, all existing WG finite element methods for Stokes problems require the stabilizing term and/or penalty term [2,10,15,16]. However, the stabilizing term and/or penalty term will add computational cost and the h^{-1} penalty factor will worsen the numerical stability for h small in solving the discrete linear system. Therefore, it is desirable to develop a WG finite element scheme without adding any stabilization/penalty term for incompressible flow.

The WG finite element method for stationary Navier–Stokes problem to be presented in this article is in the primary velocity-pressure form. To develop this method, we present a velocity-pressure weak finite element space pair $X_h \times M_h$ such that the following discrete inf-sup condition holds:

$$\sup_{\boldsymbol{v}\in X_h} \frac{(\mathrm{div}_w \boldsymbol{v}, q_h)_h}{\|\nabla_w \boldsymbol{v}\|_h} \geq \beta \|q_h\|, \ \forall\, q_h \in M_h,$$

where div_w and ∇_w are the weak divergence and weak gradient, respectively. In order to treat the nonlinear term, we establish a discrete embedding inequality on the weak finite element space. It is our belief that this embedding inequality has the potential to be useful in the analysis of WG finite element methods for other nonlinear problems. The discrete inf-sup condition and the embedding inequality enable us to show the unique existence and stability for the discrete velocity and pressure. Then, we derive the optimal error estimates for the velocity approximation in the discrete H^1-norm and pressure approximation in the L_2-norm, respectively. We emphasize that this article is the first to propose and analyze a WG finite element method for stationary Navier–Stokes problem and our techniques are also potentially applicable to other nonlinear problems.

This paper is organized as follows. In Section 2, we first recall the concepts of weak functions, their weak partial derivatives, weak gradient and weak divergence. Then we present a WG finite element scheme for the Navier–Stokes equations. Section 3 is devoted to the stability analysis for the discrete solutions. In section 4, the optimal error estimates are derived for velocity and pressure approximations, respectively. In Section 5, we provide some numerical examples to illustrate our theoretical analysis. Some conclusions are given in Section 6.

Throughout this paper, for a non-negative integer m, we adopt the notations $W^{m,r}(D)$ to denote the usual Sobolev spaces on a domain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,r,D}$ and semi-norm $|\cdot|_{m,r,D}$, and if $r = 2$, we set $W^{m,r}(D) = H^m(D)$, $\|\cdot\|_{m,r,D} = \|\cdot\|_{m,D}$. When $D = \Omega$, we omit the index D. The notations $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner product and norm in the space $L_2(\Omega)$, respectively. We will use the letter C to represent a generic positive constant, independent of the mesh size h.

2. Problem and its weak Galerkin finite element approximation

Consider the stationary Navier–Stokes equations

$$-\nu\triangle\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad in\ \ \Omega, \tag{1}$$

$$\mathrm{div}\,\boldsymbol{u} = 0, \quad in\ \ \Omega, \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{0}, \quad on\ \ \partial\Omega, \tag{3}$$