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# Perturbed normalizers and Melnikov functions ☆

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ABSTRACT

Given $m \geq 1$ and a smooth family of planar vector fields $(X_\varepsilon)_\varepsilon$ that is a perturbation of a period annulus, we provide a characterization, in terms of Lie brackets, of the property that the first $(m-1)$ Melnikov functions of $(X_\varepsilon)_\varepsilon$ vanish identically. The equivalent condition is the existence of a smooth family of planar vector fields $(U_\varepsilon)_\varepsilon$, called here *perturbed normalizers of order m*. We also provide an effective procedure for computing $U_\varepsilon$ when the first $(m-1)$ Melnikov functions of $(X_\varepsilon)_\varepsilon$ vanish identically. A formula for the derivative of the $m$-th order Melnikov function is given.



## 1. Introduction

An important problem in the theory of planar vector fields is the existence and stability of limit cycles. One of the situations when limit cycles appear is when a continuum of closed orbits is perturbed. The classical tool used in this situation is the family (depending on the perturbation parameter) of Poincaré return maps and the corresponding Melnikov functions. When dealing with polynomial planar vector fields, this problem is sometimes called Hilbert 16th problem in a period annulus [4], as a part of the celebrated and still unsolved Hilbert 16th problem. When studying the family of Poincaré maps, it is important to know which the order of the first non-null Melnikov function is. Here we provide an alternative way to find this order, which involves a family of vector fields called *perturbed normalizers* for the family of vector fields perturbing the period annulus. By the end of this section we will state precisely the main result.

A key ingredient in this work is the classical Lie bracket, which, for the smooth planar vector fields $X$ and $U$ is defined as

$$[X, U] = DX\,U - DU\,X,$$

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where $DX$ and $DU$ denote the Jacobian matrices of $X$ and $U$, respectively. Note that the following notations are also in use: $DX\,U = \partial_X U$ is the directional derivative of $U$ with respect to $X$, $[X,U] = L_X(U)$ is the Lie derivative of $U$ with respect to $X$. Recall that a vector field $U$ is said to be *transversal* to $X$ on an open set $\Omega \subset \mathbb{R}^2$ when $X \wedge U \neq 0$ on $\Omega$. The wedge product $\wedge$ between two planar vectors is the determinant of the matrix which has them as columns.

Given two transversal smooth planar vector fields $X$ and $U$ in $\Omega$ we have

$$[X,U] = \alpha\, X + \beta\, U, \text{ where } \alpha = \frac{[X,U] \wedge U}{X \wedge U} \text{ and } \beta = \frac{X \wedge [X,U]}{X \wedge U} \text{ in } \Omega.$$

Several authors, like Pleshkan, Villarini, Sabatini, Freire–Gasull–Guillamon, used the Lie bracket to study the isochronicity of a period annulus of $X$; see [5,10,11,14]. They either assumed the existence of a transversal *commutator* $U$ of $X$ (i.e. $[X,U] = 0$) in [11], or of a transversal *normalizer* $U$ of $X$ (i.e. $[X,U] = \alpha\, X$) in [5]. Afterwards, formulas for the derivatives of the period function were found and these permitted them to study other properties (e.g. monotonicity or critical points) of the period function; see [6,12,13]. The formulas were also applied to families of vector fields having a period annulus, thus obtaining results on bifurcation of critical periods; see [8,9]. Using again the Lie bracket, in [7] the authors obtained a formula for the derivative of the Poincaré return map defined in a neighborhood of a limit cycle of some vector field. They used this formula to study the stability and hyperbolicity of limit cycles, the isochrons of limit cycles and non-uniqueness of limit cycles.

Note that the formula for the derivative of the period function of $X$ (in its period annulus) was given assuming the existence of $U$ such that $[X,U] = \alpha\, X$, while the formula for the derivative of the Poincaré return map of $X$ (in a neighborhood of its limit cycle) was given assuming the existence of $U$ such that $[X,U] = \beta\, U$. A first important result of this paper gives formulas for the derivative of both the Poincaré return map and the time return map of $X$ (in a region where the orbits return to some transversal section) assuming the existence of $U$ which is only transversal to $X$, thus generalizing (or putting together) Theorem 1 in [6], Theorem 2 in [7] and Theorem 1 in [12]. This result will be proved in Section 2 as Theorem 3 and it will be used for families of planar vector fields which are perturbations of a period annulus. In fact this is our main object of study. More exactly, here we consider a $C^\infty$ smooth family $(X_\varepsilon)_\varepsilon$ of planar vector fields of the form

$$X_\varepsilon(x) = X_0(x) + \varepsilon X_1(x) + \cdots + \varepsilon^n X_n(x) + \mathcal{O}(\varepsilon^{n+1}), \quad x \in \mathbb{R}^2, \quad |\varepsilon| \ll 1,$$

for some integer $n \geq 1$. We always assume that the unperturbed vector field $X_0$ has a period annulus, that is, an open and connected region in the plane filled with non-trivial closed orbits. Let $\mathcal{P} \subset \mathbb{R}^2$ be a nonempty, open and connected subset of $\mathbb{R}^2$, such that $\overline{\mathcal{P}}$ is contained in the period annulus of $X_0$ and

$$X_0(x) \neq 0 \;\text{ for all }\; x \in \overline{\mathcal{P}}.$$

An important property of $X_0$ which will be used here is that

**Lemma 1.** *$X_0$ has a smooth transversal normalizer in $\mathcal{P}$.*

This lemma was proved by Algaba–Freire–Gamero in [1]; see also [12]. In fact, using Theorem 3 of the present paper, one can see that, *a vector field $X$ whose orbits in a region $\mathcal{P}$ return to some transversal section, has a smooth transversal normalizer in $\mathcal{P}$ if and only if all its orbits in $\mathcal{P}$ are closed.* Note that similar comments were made in [7].

Since we will work with the Melnikov functions associated to $(X_\varepsilon)_\varepsilon$, we recall their definition now. Considering an analytic transversal section $\Sigma = \{\gamma(s) \;:\; s \in I\}$ to the flow of $X_0$ in $\mathcal{P}$, we have that, for $|\varepsilon| \ll 1$, $\Sigma$ is also a transversal section to the flow of $X_\varepsilon$ in $\mathcal{P}$. We denote by