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Doubly paradoxical functions of one variable

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ABSTRACT

This paper concerns three kinds of seemingly paradoxical real valued functions of one variable. The first two, defined on $\mathbb{R}$, are the celebrated continuous nowhere differentiable functions, known as Weierstrass's monsters, and everywhere differentiable nowhere monotone functions—simultaneously smooth and very rugged—to which we will refer as differentiable monsters. The third kind was discovered only recently and consists of differentiable functions $\mathfrak{f}$ defined on a compact perfect subset $\mathfrak{X}$ of $\mathbb{R}$ which has derivative equal zero on its entire domain, making it everywhere pointwise contractive, while, counterintuitively, $\mathfrak{f}$ maps $\mathfrak{X}$ *onto* itself. The goal of this note is to show that this pointwise shrinking globally stable map $\mathfrak{f}$ can be extended to functions $f, g\colon \mathbb{R} \to \mathbb{R}$ which are differentiable and Weierstrass's monsters, respectively. Thus, we pack three paradoxical examples into two functions. The construction of f is based on the following variant of Jarník's Extension Theorem: *For every differentiable function f from a closed $P \subseteq \mathbb{R}$ into $\mathbb{R}$ there exists its differentiable extension $\hat{f}\colon \mathbb{R} \to \mathbb{R}$ such that $\hat{f}$ is nowhere monotone on $\mathbb{R} \setminus P$.*

1. Background

The number of counterintuitive examples that are known in mathematical analysis is very large, see e.g. book [9]. However, few have as much interesting history as *Weierstrass's monsters*—everywhere continuous nowhere differentiable functions from $[a, b]$ to $\mathbb{R}$—and *differentiable monsters*—the maps from $[a, b]$ to $\mathbb{R}$ that are everywhere differentiable but monotone on no interval. Shortly, the first published example of Weierstrass's monster was given by K. Weierstrass and appeared in the 1872 paper, see [7] or [22]. At that time, mathematicians commonly believed that a continuous function must have a derivative at a "significant" set of points. Thus, the example was received with disbelief and such functions eventually became known as *Weierstrass's monsters*. One of the most elegant examples of such maps comes from the 1930 paper [21] of van der Waerden. It can be defined, on $\mathbb{R}$, as

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$$f(x) \coloneqq \sum_{n=0}^{\infty} 4^n f_n(x), \tag{1}$$

where $f_n(x) \coloneqq \min_{k\in\mathbb{Z}} \left|x - \frac{k}{8^n}\right|$ is the distance from $x \in \mathbb{R}$ to the set $\frac{1}{8^n}\mathbb{Z} = \left\{\frac{k}{8^n} : k \in \mathbb{Z}\right\}$. (See [4] or [19, thm. 7.18].) A large number of simple constructions of Weierstrass's monsters can be also found in [20] or a recent book [10].

The history of differentiable monsters is described in detail in the 1983 paper of A. M. Bruckner [2]. The first construction of such a function was given in 1887 by A. Köpcke [13]. (A gap in [13] was corrected in [14,15].) The most influential study of this subject is the 1915 paper [6] of A. Denjoy. Two relatively simple constructions of differentiable monsters come from the 1970s papers [12,23]. A considerably simpler construction was recently found by the first author [4]. Specifically, a differentiable monster in [4] is defined on $\mathbb{R}$ as

$$f(x) \coloneqq h(x - t) - h(x), \tag{2}$$

where h a strictly increasing differentiable function from $\mathbb{R}$ onto $\mathbb{R}$ for which $G \coloneqq \{x \in \mathbb{R} \colon f'(x) = 0\}$ contains a countable dense set[1] D and t is chosen from a dense G_δ-set $\bigcap_{d\in D}\big((-d + G) \cap (d - G)\big)$.

The third paradoxical example we consider was first constructed in the 2016 paper [5] of the first author and J. Jasinski. Since then, the construction was further generalized, in [1], and simplified, see [4]. The example is a differentiable self-homeomorphism $\mathfrak{f}$ of a compact perfect subset $\mathfrak{X}$ of $\mathbb{R}$ with $\mathfrak{f}'(x) = 0$ for all $x \in \mathfrak{X}$. Thus, $\mathfrak{f}$ is shrinking at every $x \in \mathfrak{X}$ and so, one would expect that the diameter of $\mathfrak{f}[\mathfrak{X}]$ should be smaller than that of $\mathfrak{X}$, which evidently is not the case. Of course, $\mathfrak{X}$ must have Lebesgue measure zero, since $\mathfrak{f}' \equiv 0$ implies that $\mathfrak{f}[\mathfrak{X}]$ must have measure zero, see for example [8, p. 355]. The construction of $\mathfrak{f}$ from [4] is also simple enough to be described in few lines. Specifically, it can be defined as

$$\mathfrak{f} \coloneqq h \circ \sigma \circ h^{-1} \tag{3}$$

from $\mathfrak{X} \coloneqq h[2^\omega]$ onto itself, where $\sigma \colon 2^\omega \to 2^\omega$ is the add-one-and-carry adding machine,[2] while $h(s) \coloneqq \sum_{n=0}^{\infty} 2s_n 3^{-(n+1)N(s\restriction n)}$, while $N(s \restriction n)$ is defined as $N(s \restriction n) \coloneqq \sum_{i<n-1} s_i 2^i + (1 - s_{n-1})2^{n-1} + 2^n$. Notice, that $\mathfrak{X} = h[2^\omega]$ is a subset of the Cantor ternary set, denoted in what follows as $\mathfrak{C}$.

To extend $\mathfrak{f}$ to a differentiable monster, we will use the following result, that was first proved in the 1923 paper [11] of V. Jarník and independently rediscovered in the 1974 paper [17] of G. Petruska and M. Laczkovich. Its simplified proof, as well as a history of this result, can be found in a recent paper [3] of M. Ciesielska and the first author.

Proposition 1. (Jarník's Extension Theorem) *Every differentiable function $f \colon P \to \mathbb{R}$, where $P \subseteq \mathbb{R}$ is closed, admits a differentiable extension $\bar{f} \colon \mathbb{R} \to \mathbb{R}$.*

The differentiability of $f \colon P \to \mathbb{R}$ is understood as the existence of its derivative, that is, a function $f' \colon P' \to \mathbb{R}$ where $P' \subseteq P$ is the set of all accumulation points of P and $f'(p) \coloneqq \lim_{x\to p,\ x\in P} \frac{f(x)-f(p)}{x-p}$ for every $p \in P'$.

In our proof we will also use the following well known result.

Proposition 2. (Folklore) *For every closed $K \subseteq \mathbb{R}$, there exists a C^∞ function $g \colon \mathbb{R} \to \mathbb{R}$ such that $g(x) = g'(x) = 0$ for all $x \in K$ and $g(x) > 0$ on $K^c = \mathbb{R} \setminus K$.*

[1] Such a map h was first constructed in the 1907 paper [18] of D. Pompeiu. It can be defined as the inverse of a function $g(x) \coloneqq \sum_{i=1}^{\infty} 2^{-i}(x - q_i)^{1/3}$, where $\{q_i : i \in \mathbb{N}\}$ is an enumeration of rational numbers such that $|q_i| \le i$ for all $i \in \mathbb{N}$.

[2] For $s = \langle s_0, s_1, s_2, \dots\rangle \in 2^\omega$ it is defined: $\sigma(s) \coloneqq \langle 0, 0, 0, \dots\rangle$ when $s_i = 1$ for all $i < \omega$ and, otherwise, $\sigma(s) \coloneqq \langle 0, 0, \dots, 0, 1, s_{k+1}, s_{k+2}, \dots\rangle$, where $s_k = 0$ and $s_i = 1$ for all $i < k$.