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Simple matrix representations of the orthogonal polynomials for a rational spectral density on the unit circle

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ABSTRACT

In this note, by using a discrete analog of a projection formula introduced by A. Seghier in 1978, we calculate the orthogonal polynomials on the unit circle for a rational spectral density having no zeros there, and derive simple matrix representations of themselves, their squared norms, and the Verblunsky coefficients.

1. Introduction

Let μ be a probability measure on the unit circle $\mathbb{T}$ that is supported on an infinite set. By applying the Gram–Schmidt method to the sequence of monomials $1, z, z^2, \ldots$ in the Hilbert space $L^2(\mu)$ with

$$(\varphi, \psi)_\mu = \int_0^{2\pi} \varphi(e^{i\theta})\overline{\psi(e^{i\theta})}d\mu(\theta), \qquad \|\varphi\|_\mu = \left(\int_0^{2\pi} |\varphi(e^{i\theta})|^2 d\mu(\theta)\right)^{1/2},$$

one obtains a sequence of the *monic orthogonal polynomials* $\Phi_0, \Phi_1, \Phi_2, \ldots$ with respect to μ, i.e.,

$$\Phi_n(z) = z^n + \text{lower order}, \qquad (\Phi_n, \Phi_{n'})_\mu = 0 \qquad (n \neq n').$$

Their constant terms $\alpha_n = \Phi_n(0)$ lie in the open unit disc $\mathbb{D}$ for all $n = 1, 2, \ldots$. Therefore, μ yields a sequence $\{\alpha_1, \alpha_2, \ldots\}$ in $\mathbb{D}$. Verblunsky's theorem states that every sequence in $\mathbb{D}$ arises in this way from a unique probability measure on $\mathbb{T}$ that is supported on an infinite set, whence α_n are called the

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Verblunsky coefficients. (There are several other names such as the Schur parameters.) As an application, the monic orthogonal polynomials are used for studying the prediction problem for a centered, complex-valued stationary time series $\{X_k\}$ with spectral measure μ, i.e.,

$$\mathrm{Cov}(X_j, X_k) = \int_0^{2\pi} e^{i(j-k)\theta} d\mu(\theta) \qquad (j, k = 0, \pm 1, \pm 2, \ldots).$$

More precisely, Φ_n is a spectral counterpart of the one-step prediction error in the sense that

$$X_n - \hat{X}_n = X_n - \sum_{k=1}^{n} \phi_{n,k} X_{n-k} \quad \leftrightarrow \quad \Phi_n(z) = z^n - \sum_{k=1}^{n} \phi_{n,k} z^{n-k}, \qquad \mathrm{Var}(X_n - \hat{X}_n) = \|\Phi_n\|_\mu^2,$$

where $\hat{X}_n$ is the linear predictor of X_n based on $\{X_0, X_1, \ldots, X_{n-1}\}$. Since $\phi_{n,n} = -\alpha_n$, the Verblunsky coefficients are essentially the same as the *partial autocorrelation function* $\{\phi_{n,n}\}$. See Simon [11] for the theory of orthogonal polynomials on the unit circle, and Brockwell–Davis [2] for the prediction problem of stationary time series.

Inoue [3–5], Inoue–Kasahara [6,7] and Bingham et al. [1] have developed explicit representations of the predictor coefficients and prediction error variances, and studied their applications. Taking a different approach, this note presents simple matrix representations of the monic orthogonal polynomials and related quantities for a probability measure of the form $d\mu(\theta) = w(e^{i\theta})\frac{d\theta}{2\pi}$, where w is a rational function of $e^{i\theta}$ satisfying $w(e^{i\theta}) \neq 0$ on $\mathbb{T}$. As is well known, the latter can be expressed as

$$w(e^{i\theta}) = \frac{|P(e^{i\theta})|^2}{|Q(e^{i\theta})|^2} \tag{1}$$

with some polynomials P and Q which have no common zeros and satisfy

$$P(0) > 0, \qquad Q(0) > 0, \qquad P(z) \neq 0, \qquad Q(z) \neq 0 \qquad (z \in \mathbb{D} \cup \mathbb{T}). \tag{2}$$

Such a rational function is important in time series analysis since it is the spectral density of a causal and invertible ARMA (autoregressive-moving average) process, characterized by the difference equation

$$Q(B)X_k = P(B)Z_k \qquad (k = 0, \pm 1, \pm 2, \ldots),$$

where B is the backward shift operator ($BX_k = X_{k-1}$) and $\{Z_k\}$ is a white noise ($\mathrm{Cov}(Z_j, Z_k) = \delta_{jk}$). The matrix representations are derived by calculating Φ_n via an analog of a projection formula introduced by Seghier [10], which is explained in Section 2. The representations themselves, including those of the squared norms $\|\Phi_n\|_\mu^2$ and the Verblunsky coefficients α_n, are given in Section 3. Among other results, the representation of α_n is particularly simple.

We dedicate this note to Yusuke Nakamura (1992–2016) who discovered a similar matrix representation of the finite predictor for some continuous-time process, in joint work with the first author. It was this discovery that made the latter realize, after his death, the existence of explicit formulas in a simple case of this paper, and thus led us to the present work.

2. Seghier's formula

In his work on the prediction problem for a continuous-time stationary process, Seghier [10] introduced a projection formula which is useful when the process has a rational spectral density. This section is devoted to a brief discussion on its discrete analog and related matters. For details, see Kasahara–Bingham [8].

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