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Four-body central configurations with one pair of opposite sides parallel

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ABSTRACT

We study four-body central configurations with one pair of opposite sides parallel. We use a novel constraint to write the central configuration equations in this special case, using distances as variables. We prove that, for a given ordering of the mutual distances, a trapezoidal central configuration must have a certain partial ordering of the masses. We also show that if opposite masses of a four-body trapezoidal central configuration are equal, then the configuration has a line of symmetry and it must be a kite. In contrast to the general four-body case, we show that if the two adjacent masses bounding the shortest side are equal, then the configuration must be an isosceles trapezoid, and the remaining two masses must also be equal.

Contents

1. Introduction

Let P_1, P_2, P_3, and P_4 be four points in $\mathbb{R}^3$ with position vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$, and $\mathbf{q}_4$, respectively. Let $r_{ij} = \|\mathbf{q}_i - \mathbf{q}_j\|$, be the distance between the point P_i and P_j, and let $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \in \mathbb{R}^{12}$. The center of mass of the system is $\mathbf{q}_{CM} = \frac{1}{M}\sum_{i=1}^n m_i \mathbf{q}_i$, where $M = m_1 + \ldots m_n$ is the total mass. The Newtonian 4-body problem concerns the motion of 4 particles with masses $m_i \in \mathbb{R}^+$ and positions $\mathbf{q}_i \in \mathbb{R}^3$, where $i = 1, \ldots, 4$. The motion is governed by Newton's law of motion

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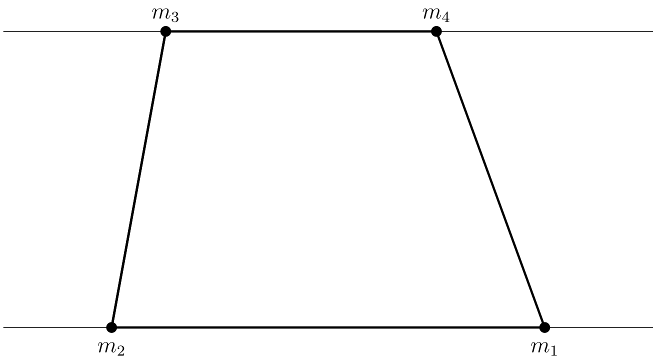

Fig. 1. An example of a trapezoidal central configuration.

$$m_i\ddot{\mathbf{q}}_i = \sum_{i\neq j} \frac{m_i m_j(\mathbf{q}_j - \mathbf{q}_i)}{r_{ij}^3} = \frac{\partial U}{\partial \mathbf{q}_i}, \quad 1 \leq i \leq 4 \tag{1}$$

where $U(\mathbf{q})$ is the Newtonian potential

$$U(\mathbf{q}) = \sum_{i<j} \frac{m_i m_j}{r_{ij}}, \quad 1 \leq i \leq 4. \tag{2}$$

A *central configuration* (c.c.) of the four-body problem is a configuration $\mathbf{q} \in \mathbb{R}^{12}$ which satisfies the algebraic equations

$$\lambda\, m_i(\mathbf{q}_i - \mathbf{q}_{CM}) = \sum_{i\neq j} \frac{m_i m_j(\mathbf{q}_j - \mathbf{q}_i)}{r_{ij}^3}, \quad 1 \leq i \leq n. \tag{3}$$

If we let $I(\mathbf{q})$ denote the moment of inertia, that is,

$$I(\mathbf{q}) = \frac{1}{2}\sum_{i=1}^{n} m_i \|\mathbf{q}_i - \mathbf{q}_{CM}\|^2 = \frac{1}{2M}\sum_{1\leq i<j\leq n} m_i m_j r_{ij}^2,$$

we can write equations (3) as

$$\nabla U(\mathbf{q}) = \lambda \nabla I(\mathbf{q}). \tag{4}$$

Viewing λ as a Lagrange multiplier, a central configuration is simply a critical point of U subject to the constraint I equals a constant.

A central configuration is *planar* if the four points P_1, P_2, P_3, and P_4 lie on the same plane. Equations (3), and (4) also describe planar central configurations provided $\mathbf{q}_i \in \mathbb{R}^2$ for $i = 1, \ldots 4$. We say that a planar configuration is *degenerate* if two or more points coincide, or if more than two points lie on the same line. Non-degenerate planar configurations can be classified as either *concave* or *convex*. A concave configuration has one point which is located strictly inside the convex hull of the other three, whereas a convex configuration does not have a point contained in the convex hull of the other three points. Any convex configuration determines a convex quadrilateral (for a precise definition of quadrilateral see for example [5]). In a planar convex configuration we say that the points are *ordered sequentially* if they are numbered consecutively while traversing the boundary of the corresponding convex quadrilateral. In this paper we are interested in studying *trapezoidal central configurations*, that is, those c.c.'s for which two of the opposite sides are parallel (see Fig. 1). Non-degenerate trapezoidal central configurations are necessarily convex.