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A comparison theorem for two divided differences and applications to special functions

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ABSTRACT

In this paper, we present a general comparison theorem for two divided differences of a three times differentiable function. This gives a unified treatment for (logarithmically) complete monotonicity, monotonicity and inequalities involving some special functions including gamma, psi and polygamma functions. As their consequences, we not only refine and generalize some important results, but also present simple and interesting alternative proofs of certain earlier results.

1. Introduction

The Euler's gamma and psi (digamma) functions are defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively. The derivatives ψ', ψ'', ψ''', ... are known as polygamma functions.

Denote by $\psi_n = (-1)^{n-1} \psi^{(n)}$ for $n \in \mathbb{N}$ and $\psi_{-1} = \ln \Gamma$, $\psi_0 = -\psi$. Then ψ_n has some simple properties:

(P1) $\psi_n' = -\psi_{n+1} < 0$ for $n \geq 0$.
(P2) ψ' is strictly completely monotonic on $(0, \infty)$, and so is ψ_n for $n \geq 0$.
(P3) The sequence $\{\psi_{n+1}/\psi_n\}_{n\in\mathbb{N}}$ is strictly increasing and concave.

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(P4) ψ_n/ψ_{n+1} for $n \in \mathbb{N}$ is strictly increasing and convex on $(0, \infty)$.
(P5) ψ_n for $n \in \mathbb{N}$ is log-convex on $(0, \infty)$.
(P6) $x\psi_{n+1}/\psi_n$ for $n \in \mathbb{N}$ is strictly decreasing from $(0, \infty)$ onto $(n, n + 1)$.
(P7) $\psi_{n+1}^2/(\psi_n\psi_{n+2})$ is strictly decreasing from $(0, \infty)$ onto $(n/(n + 1), (n + 1)/(n + 2))$.

Properties (P3)–(P5) were proved in [41,47], (P6) is due to Alzer [4,5], while (P7) was showed in [41]. More properties of polygamma functions can be found in [2–6,41], [8, Theorem 2.7], [21,33,38,40].

Let $f : I \to \mathbb{R}$ be (strictly) monotonic and $a, b \in I$. Then the so-called integral f-mean of a and b is defined as [16]

$$\mathcal{I}_f(a, b) = f^{-1}\left(\frac{\int_a^b f(x)\,dx}{b-a}\right) \text{ if } a \neq b \text{ and } \mathcal{I}_f(a, a) = a.$$

Elezović and Pečarić [16, Theorem 6] proved that for $a, b > 0$, $\mathcal{I}_{\psi'}(a, b) \le \mathcal{I}_\psi(a, b)$ and the function $x \mapsto \mathcal{I}_\psi(x + a, x + b) - x$ is increasing concave with

$$\lim_{x\to\infty} (\mathcal{I}_\psi(x + a, x + b) - x) = \frac{a+b}{2}.$$

And therefore, for $a, b > 0$ the double inequality

$$x + \mathcal{I}_\psi(a, b) < \mathcal{I}_\psi(x + a, x + b) < x + \frac{a+b}{2}$$

holds for $x \ge 0$. Very recently, Yang and Zheng [47] further showed that for $a, b > 0$ with $a \neq b$, the sequence $\{\mathcal{I}_{\psi_n}(a, b)\}_{n\ge0}$ is strictly decreasing with $\lim_{n\to\infty} \mathcal{I}_{\psi_n}(a, b) = \min(a, b)$, and the function $x \mapsto A_{\psi_n}(x) = \mathcal{I}_{\psi_n}(x + a, x + b) - x$ is strictly increasing from $(-\min(a, b), \infty)$ onto $(\min(a, b), (a + b)/2)$. And consequently, the double inequality

$$x + \min(a, b) < \mathcal{I}_{\psi_n}(x + a, x + b) < x + \frac{a+b}{2}$$

holds for all $x > -\min(a, b)$.

In [3, Theorem 2] Alzer established that for an integer $n \ge 0$ and a real number $s \in (0, 1)$, the double inequality

$$\frac{n!}{(x + \alpha_n(s))^{n+1}} < \frac{\psi_n(x + s) - \psi_n(x + 1)}{1 - s} < \frac{n!}{(x + \beta_n(s))^{n+1}} \tag{1.1}$$

holds for all real numbers $x > 0$ with the best possible constants

$$\alpha_n(s) = \left(\frac{\psi_n(s) - \psi_n(1)}{n!(1 - s)}\right)^{-1/(n+1)} \quad \text{and} \quad \beta_n(s) = \frac{s}{2}.$$

If let $s \to 1$ then inequality (1.1) becomes as

$$\frac{n!}{[x + \alpha_n(1)]^{n+1}} < \left|\psi^{(n)}(x + 1)\right| < \frac{n!}{[x + \beta_n(1)]^{n+1}}, \tag{1.2}$$

where $\alpha_n(1) = \left(n!/\left|\psi^{(n+1)}(1)\right|\right)^{1/(n+1)}$ and $\beta_n(1) = 1/2$ are the best constants, which was proved in [21, Theorem 1] by Guo, Qi, Zhao and Luo.