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Resolvent expansion for the Schrödinger operator on a graph with infinite rays

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ABSTRACT

We consider the Schrödinger operator on a combinatorial graph consisting of a finite graph and a finite number of discrete half-lines, all jointed together, and compute an asymptotic expansion of its resolvent around the threshold 0. Precise expressions are obtained for the first few coefficients of the expansion in terms of the generalized eigenfunctions. This result justifies the classification of threshold types solely by growth properties of the generalized eigenfunctions. By choosing an appropriate free operator a priori possessing no zero eigenvalue or zero resonance we can simplify the expansion procedure as much as that on the single discrete half-line.

Contents

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1. Introduction

The purpose of this paper is to compute an asymptotic expansion of the resolvent around the threshold 0 for the discrete Schrödinger operator

$$H = -\Delta_G + V \tag{1.1}$$

on an infinite, undirected and simple graph. Here we denote the set of vertices by G, and the set of edges by E_G, hence we are considering the graph (G, E_G). We sometimes call it simply the graph G. For any function $u\colon G \to \mathbb{C}$ the Laplacian $-\Delta_G$ is defined as

$$(-\Delta_G u)[x] = \sum_{y\sim x}(u[x] - u[y]), \tag{1.2}$$

where for any two vertices $x, y \in G$ we say $x \sim y$ if $\{x, y\} \in E_G$. We assume that the graph G consists of a finite graph K and a finite number of discrete half-lines L_α, $\alpha = 1, \dots, N$, jointed together. Special cases are the discrete full line $\mathbb{Z}$ and the discrete half-line $\mathbb{N}$, considered in [10] and [11], respectively. The perturbation V can be a general non-local operator, which is assumed to decay at infinity in an appropriate sense.

The main results in this paper give a complete description in terms of growth properties of the generalized eigenfunctions for the first few coefficients of the resolvent expansion:

$$(H + \kappa^2)^{-1} = \kappa^{-2}G_{-2} + \kappa^{-1}G_{-1} + G_0 + \kappa G_1 + \cdots .$$

More precisely, we prove that G_{-2} is the *bound projection* or the projection onto the bound eigenspace, and that G_{-1} is the *resonance projection* or the projection onto the resonance eigenspace. Explicit expressions for G_0 and G_1 are also computed. It is well known that the coefficients G_{-2} and G_{-1} directly affect the local decay rate of the Schrödinger propagator e^{-itH} as $t \to \pm\infty$, see [12]. Hence our results reveal a relationship between the growth rates of the generalized eigenfunctions in space and the local decay rate of e^{-itH} in time, justifying a classification of threshold types solely by the structure of the generalized eigenspace.

There exists a large literature on threshold resolvent expansions for Schrödinger operators. However, a complete analysis taking into account all possible generalized threshold eigenfunctions has been obtained only recently. The first one is in [10] on the discrete full line $\mathbb{Z}$ and more recently on the discrete half-line $\mathbb{N}$ in [11]. In these papers the authors implement the expansion scheme of [13,14] in its full generality.

This paper is a generalization of [10,11] to a graph with infinite rays. The strategy is essentially the same as before. However, in this paper, based on ideas from [4] and [11], we set up a free operator a priori possessing no zero eigenvalue or zero resonance, and this effectively simplifies the expansion procedure to