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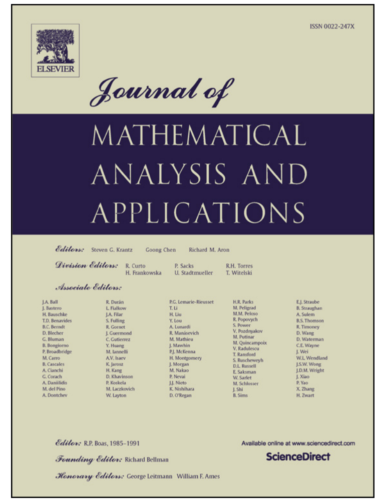

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Normal Structure and Orbital Fixed Point Conditions

W. A. Kirk and Naseer Shahzad

Abstract. In some sense all of the results discussed in this paper are an outgrowth of the concept of 'normal structure' introduced by Brodskii and Milman in 1948 and following related idea introduced by Belluce and Kirk in 1969. A mapping T of a metric space K into itself is said to have 'diminishing orbital diameters' if given any $x \in K$ it is the case that

$$\lim_{n\to\infty} diam\left(O\left(T^n\left(x\right)\right)\right) < diam\left(O\left(x\right)\right)$$

whenever $diam\left(O\left(x\right)\right) > 0$, where $O\left(x\right) = \left\{x, T\left(x\right), T^2\left(x\right), \cdots\right\}$. Among other things Belluce and Kirk observed that the assumption of diminishing orbital diameters on a mapping $T : K \to K$ is sufficient to assure that T has a fixed point if T is nonexpansive and if K is a weakly compact convex subset of a Banach space. It was later shown by Kirk that the convexity assumption on K can be dropped.

The purpose of this discussion is to illustrate how the above ideas are related to a number of similar ones that have been introduced more recently. Among these is the following concept introduced recently by Amini-Harandi, et al. A mapping $T : K \to K$ of a subset K of a Banach space is said to be *nonexpansive wrt orbits* if for all $x, y \in K$,

$$\|T\left(x\right) - T\left(y\right)\| \leq r_x\left(O\left(y\right)\right)$$

where

$$r_x\left(O\left(y\right)\right) = \sup\left\{\|x-u\| : u \in O\left(y\right)\right\}.$$

It is shown that if K is a weakly compact subset of a Banach space and if $T : K \to K$ is nonexpansive wrt orbits then the following condition always assures the existence of a fixed point for T : For each $x \in K$ with $T\left(x\right) \neq x$,

$$\inf_{m\in\mathbb{N}}\left\{\lim\sup_{n\to\infty} \|T^m\left(x\right) - T^n\left(x\right)\|\right\} < diam\left(O\left(x\right)\right).$$

Some open questions are also discussed.

1. Introduction

In this paper we discuss how certain classical metric fixed point theorems, and especially theorems involving nonexpansive mappings published over forty years ago, relate to some recently published results. As a point of departure we revisit results obtained in [**3**] and [**16**], and show how they are related to ideas that have emerged quite recently. The underlying setting is a Banach space, and the basic

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