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The stochastic order of probability measures on ordered metric spaces

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A R T I C L E I N F O

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A B S T R A C T

The general notion of a stochastic ordering is that one probability distribution is smaller than a second one if the second attaches more probability to higher values than the first. Motivated by recent work on barycentric maps on spaces of probability measures on ordered Banach spaces, we introduce and study a stochastic order on the space of probability measures $\mathcal{P}(X)$, where X is a metric space equipped with a closed partial order, and derive several useful equivalent versions of the definition. We establish the antisymmetry and closedness of the stochastic order (and hence that it is a closed partial order) for the case of a partial order on a Banach space induced by a closed normal cone with interior. We also consider order-completeness of the stochastic order for a cone of a finite-dimensional Banach space and derive a version of the arithmetic-geometric-harmonic mean inequalities in the setting of the associated probability space on positive invertible operators on a Hilbert space.

1. Introduction

The stochastic order for random variables X, Y from a probability measure space (M, P) to $\mathbb{R}$ is defined by $X \leq Y$ if $P(X > t) \leq P(Y > t)$ for all $t \in \mathbb{R}$. This notion extends directly to random variables into $\mathbb{R}^n$ equipped with the coordinatewise order. Alternatively one can define a stochastic order on the Borel probability measures on $\mathbb{R}^n$ by $\mu \leq \nu$ if for each $s \in \mathbb{R}^n$, $\mu(s < t) \leq \nu(s < t)$, where $(s < t) := \{t \in \mathbb{R}^n : s < t\}$. One then has for random variables X, Y, $X \leq Y$ in the stochastic order if and only if $P_X \leq P_Y$, where P_X, P_Y are the push-forward probability measures with respect to X, Y respectively.

There are important metric spaces which are equipped with a naturally defined partial order, for example the open cone $\mathbb{P}_n$ of positive definite matrices of some fixed dimension, where the order is the Loewner order.

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One can use the Loewner order to define an order on $\mathcal{P}(\mathbb{P}_n)$, the space of Borel probability measures, an order that we call the stochastic order, as it generalizes the case of $\mathbb{R}$ or $\mathbb{R}^n$.

In this paper we broadly generalize the stochastic order to an order on the set $\mathcal{P}(X)$ of Borel probability measures on a partially ordered metric space $(X, d, \leq)$. We develop basic properties of this order and specialize to the setting of normal cones in Banach spaces to show that the stochastic order in that setting is indeed a partial order. A special example is the cone $\mathbb{P}$ of positive invertible operators on a Hilbert space. The order on $\mathcal{P}(\mathbb{P})$ plays a crucial role in the study of means of operators that has recently been under active investigation.

The paper is organized as follows. Section 2 is a short preliminary on Borel measures. In Section 3 we give the general definition of the stochastic order on $\mathcal{P}(X)$ for a partially ordered metric space X and derive several useful alternative formulations. In Section 4 we show for normal cones with interior that the stochastic order on $\mathcal{P}(X)$ is indeed a partial order (the antisymmetry being the nontrivial property to establish). In Section 5 we show in the normal cone setting that the stochastic partial order is a closed order with respect to the weak topology, and hence with respect to the Wasserstein topology. In Section 6 we consider the order-completeness of $\mathcal{P}(X)$, and in Section 7 derive a version of the arithmetic-geometric-harmonic mean inequalities in the setting of the probability space $\mathcal{P}(\mathbb{P})$ on the positive invertible operators.

In what follows $\mathbb{R}^+ = [0, \infty)$.

2. Borel measures

In this section we recall some basic results about Borel measures on metric spaces that will be needed in what follows. As usual the Borel algebra on a metric space (X, d) is the smallest σ-algebra containing the open sets and a finite positive Borel measure is a countably additive measure μ defined on the Borel sets such that $\mu(X) < \infty$. We work exclusively with finite positive Borel measures, primarily those that are probability measures.

Recall that a Borel measure μ is *τ-additive* if $\mu(U) = \sup_\alpha \mu(U_\alpha)$ for any *directed* union $U = \bigcup_\alpha U_\alpha$ of open sets. The measure μ is said to be *inner regular* or *tight* if for any Borel set A and $\varepsilon > 0$ there exists a compact set $K \subseteq A$ such that $\mu(A) - \varepsilon < \mu(K)$. A tight finite Borel measure is also called a Radon measure.

Probability on metric spaces has been carried out primarily for separable metric spaces [3, Chapter 11], although results exist for the non-separable setting. We recall the following result, which can be more-or-less cobbled together from results in the literature; see [7] for more details.

Proposition 2.1. *A finite Borel measure μ on a metric space (X, d) has separable support. The following three conditions are equivalent*:

(1) *The support of μ has measure $\mu(X)$.*
(2) *The measure μ is τ-additive.*
(3) *The measure μ is the weak limit of a sequence of finitely supported measures.*

If in addition X is complete, these are also equivalent to:

(4) *The measure μ is inner regular.*

Proof. For a proof of separability and the equivalence of the first three conditions, see [7]. Assume that X is complete. Let μ be a finite Borel measure, and suppose (1)–(3) hold. Then the support S of μ is closed, separable and has measure 1. Let A be any Borel measurable set. Then $\mu(A \cap (X \setminus S)) = 0$ since $\mu(X \setminus S) = 0$, so $\mu(A) = \mu(A \cap S)$. Since the metric space S is a separable complete metric space, it is a standard result

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