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Higher dimensional Darboux transformations

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A B S T R A C T

We give conditions for a first order partial differential operator to serve as an n-dimensional Darboux transformation between two second-order partial differential operators. We show that some of these conditions are given by first order nonlinear partial differential equations which may be solved using the method of characteristics. We also discuss the well-known example of the Boiti–Leon–Pempinelli equation and some others.

Contents

1. Introduction

In 1882 Darboux [4] found a one-dimensional transformation for eigenfunctions and potentials of Sturm–Liouville (Schrödinger) equations with the application to the theory of surfaces. The main application of Darboux transformations in the modern theory of solitons is the efficient generation of multi-soliton solutions (see [13,5]).

The one-dimensional theory of Darboux transformations has been extensively developed (for instance see [13] and references therein). Two-dimensional Darboux transformations are considered in [5,7,8]. Other higher-dimensional generalizations have been studied by way of several approaches. One way is to consider

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factorization of multi-dimensional hamiltonians [1,2], which opens the possibility to solve equations from supersymmetric quantum mechanics. Another method based on generalizations of classical algorithms for factorization of partial differential equations has been developed in [16–19,10,11,6,5,12].

In this paper the conditions are given for a first-order single partial differential operator to serve as an n-dimensional Darboux transformation between two second-order partial differential operators. These conditions simplify the intertwining conditions which are used to characterize the Darboux transformation. We compare our results with ones obtained by other methods (see Examples 3.3, 3.4). Note that the direct construction of Darboux transformations for the Weyl first-order system using the intertwining relations is considered in [15]. We hope that work in this direction will give rise to a wider range of multi-dimensional integrable nonlinear dynamic equations.

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2. Definitions and results

Let $\mathbb{Z}_+$ denote the nonnegative integers and let $I = \mathbb{Z}_+^n$ be a set of multi-indices generated as a $\mathbb{Z}$-module by elements $e_1, \ldots, e_n$, where $(e_j)_k = \delta_{jk}$, and containing the zero element $0 := (0, \ldots, 0)$. For any $\beta \in I$, define $|\beta| = \sum_{i=1}^n \beta_i$. In what follows β will usually be our notational choice for a generic element of I.

We use I for labeling both coefficient functions and partial derivatives. To distinguish between these situations we will use superscripts to refer to derivatives; for $x = (x_1, \ldots, x_n)$ and a function $w(x)$ we define

$$\partial^\beta = \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_1} \ldots \partial_{x_n}^{\beta_n}$$

and write $w^{(\beta)}$ to mean $\partial^\beta w(x)$. By convention ∂^0 is taken to have no effect.

Let $C^k(\mathbb{R}^n)$ be the set of k-times differentiable functions with continuous kth derivatives in $\mathbb{R}^n$. Note that in applications it is often sufficient to only assume differentiability and continuity within a specific domain.

We consider two linear second-order partial differential operators

$$L[j] := \sum_{0 \le |\beta| \le 2} u_\beta[j](x) \partial^\beta, \quad j = 0, 1. \tag{2.1}$$

We assume throughout that coefficient functions $u_\beta[j](x)$ are continuously differentiable and, as we have already begun to do, will usually suppress functions' x-dependence after they have been introduced.

Amongst other things, a Darboux transformation should intertwine the operators $L[0]$ and $L[1]$. To define such a transformation we need to introduce some new operators. Firstly we define a generic "first-order directional derivative"

$$\partial^s := \sum_{|\beta|=1} d_\beta(x) \partial^\beta \tag{2.2}$$

and our (putative) Darboux transformation

$$D = \partial^s + d_0(x) = \sum_{|\beta| \le 1} d_\beta(x) \partial^\beta, \tag{2.3}$$

where the coefficient functions $d_\beta(x)$ are also assumed to be in $C^2(\mathbb{R}^n)$. We write $w^{(s)}$ for the derivative $\partial^s w$ and

$$L^{(s)}[j] := \partial^s L[j] = \sum_{|\beta| \le 2} u_\beta^{(s)}[j](x) \partial^\beta. \tag{2.4}$$