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The a-values of the Riemann zeta function near the critical line ☆

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ABSTRACT

We study the value distribution of the Riemann zeta function near the line $\operatorname{Re} s = 1/2$. We find an asymptotic formula for the number of a-values in the rectangle $1/2 + h_1/(\log T)^\theta \le \operatorname{Re} s \le 1/2 + h_2/(\log T)^\theta$, $T \le \operatorname{Im} s \le 2T$ for fixed $h_1, h_2 > 0$ and $0 < \theta < 1/13$. To prove it, we need an extension of the valid range of Lamzouri, Lester and Radziwiłł's recent results on the discrepancy between the distribution of $\zeta(s)$ and its random model. We also propose the secondary main term for the Selberg's central limit theorem by providing sharper estimates on the line $\operatorname{Re} s = 1/2 + 1/(\log T)^\theta$.

1. Introduction

The study of the Riemann zeta function $\zeta(s)$ has been one of the central topics in analytic number theory. Along with its importance in prime number theorem, various aspects of this function has been studied. Some of these studies involve the general value distribution of zeta functions in the critical strip, which has its own interests.

Let a be a nonzero complex number. The solutions to $\zeta(s) = a$, which we denote by $\rho_a = \beta_a + i\gamma_a$, are called the a-values of $\zeta(s)$. We first introduce some basic facts about them. By a general theorem of Dirichlet series, there is $A = A(a) > 0$ such that there is no a-value of $\zeta(s)$ on $\operatorname{Re} s \ge A$. There is a number $N_0(a) > 0$ such that for each $n \ge N_0(a)$, there is an a-value of $\zeta(s)$ very close to $s = -2n$ and there are at most finitely many other a-values in $\operatorname{Re} s \le 0$. (See [3] or [9].) Thus the remaining a-values lie in the strip $0 < \operatorname{Re} s < A$. For these a-values we have

$$N_a(T) := \sum_{\substack{\beta_a > 0 \\ 0 < \gamma_a < T}} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O_a(\log T)$$

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for $a \neq 1$ and

$$N_a(T) = \frac{T}{2\pi} \log \frac{T}{4\pi e} + O(\log T)$$

for $a = 1$.

Selberg observed various aspects of a-values of $\zeta(s)$. For example, he showed that at least 1/2 of the nontrivial a-values of $\zeta(s)$ lie to the left of $\operatorname{Re} s = 1/2$ assuming the Riemann hypothesis. He also conjectured that approximately 3/4 of the nontrivial a-values lie on the strip $0 < \operatorname{Re} s < 1/2$. He did not publish these ideas, however Tsang wrote them with proofs in his thesis [13, Chapter 1]. He further extended this idea and applied to a linear combination of L-functions. We refer [11] for the result and discussion in this direction.

In recent years, several studies were made in connection with values of Riemann zeta function. Gonek, Lester and Milinovich [3] showed that a positive proportion, in terms of the proportion of zeros of Riemann zeta function on the critical line, of the a-values are simple under Selberg's conjecture on the number of a-values on the strip $0 < \operatorname{Re}(s) < 1/2$ mentioned above. In connection with the linear independence conjecture (LI), Li and Radziwiłł [10] estimated the second moment of Riemann zeta function and the variants in the vertical arithmetic progression on the critical line and showed that at least one third of a given arithmetic progression are not the ordinates of the zeros. Also, Dixit, Robles, Roy and Zaharescu [2] studied the zeros of the linear combination of bounded vertical shifts of the completed Riemann zeta function and such function has infinitely many zeros on the critical line.

We now focus on the a-values to the right of 1/2-line. For fixed $1/2 < \sigma_1 < \sigma_2$, Borchsenius and Jessen [1] proved that there exists a constant $c(a, \sigma_1, \sigma_2)$ such that

$$N_a(\sigma_1, \sigma_2; T) := \sum_{\substack{\sigma_1 < \beta_a < \sigma_2 \\ 0 < \gamma_a < T}} 1 \sim c(a, \sigma_1, \sigma_2)T \tag{1}$$

as $T \to \infty$, where

$$c(a, \sigma_1, \sigma_2) = \int_{\sigma_1}^{\sigma_2} g_a(u) du$$

for some real valued function g_a with $g_a(u) > 0$ for $1/2 < u \leq 1$. Recently, Lamzouri, Lester and Radziwiłł [7] reduced the size of the error in (1) and proved that

$$N_a(\sigma_1, \sigma_2; T) = c(a, \sigma_1, \sigma_2)T + O\left(T \frac{\log\log T}{(\log T)^{\sigma_1/2}}\right)$$

holds for fixed $1/2 < \sigma_1 < \sigma_2 < 1$ and $T \geq 3$.

In this paper, we extends the above estimates near the 1/2-line.

Theorem 1.1. *Let $0 < \theta < 1/13$ be fixed, $T \geq 3$ and*

$$\sigma_T := \sigma_T(\theta) = \frac{1}{2} + \frac{1}{(\log T)^\theta}.$$

Then

$$N_a(\sigma_T; T, 2T) := \sum_{\substack{\sigma_T < \beta_a \\ T < \gamma_a < 2T}} 1 = \frac{T(\log T)^\theta}{8\pi^{3/2}\sqrt{\theta}\sqrt{\log\log T}} + O\left(T \frac{(\log T)^\theta}{(\log\log T)^{3/4}}\right).$$