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# Parabolic–elliptic chemotaxis model with space–time dependent logistic sources on $\mathbb{R}^N$. II. Existence, uniqueness, and stability of strictly positive entire solutions

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A B S T R A C T

The current work is the second of the series of three papers devoted to the study of asymptotic dynamics in the following parabolic–elliptic chemotaxis system with space and time dependent logistic source,

$$\begin{cases} \partial_t u = \Delta u - \chi \nabla \cdot (u \nabla v) + u(a(x,t) - ub(x,t)), & x \in \mathbb{R}^N, \\ 0 = \Delta v - \lambda v + \mu u, & x \in \mathbb{R}^N, \end{cases} \tag{0.1}$$

where $N \geq 1$ is a positive integer, $\chi$, $\lambda$ and $\mu$ are positive constants, and the functions $a(x,t)$ and $b(x,t)$ are positive and bounded. In the first of the series, we studied the phenomena of pointwise and uniform persistence, and asymptotic spreading in (0.1) for solutions with compactly supported or front like initials. In the second of the series, we investigate the existence, uniqueness and stability of strictly positive entire solutions of (0.1). In this direction, we prove that, if $0 \leq \mu\chi < \inf_{x,t} b(x,t)$, then (0.1) has a strictly positive entire solution, which is time-periodic (respectively time homogeneous) when the logistic source function is time-periodic (respectively time homogeneous). Next, we show that there is positive constant $\chi_0$, depending on $N$, $\lambda$, $\mu$, $a$ and $b$ such that for every $0 \leq \chi < \chi_0$, (0.1) has a unique positive entire solution which is uniform and exponentially stable with respect to strictly positive perturbations. In particular, we prove that $\chi_0$ can be taken to be $\inf_{x,t} \frac{b(x,t)}{2\mu}$ when the logistic source function is either space homogeneous or the function $(x,t) \mapsto \frac{b(x,t)}{a(x,t)}$ is constant. We also investigate the disturbances to Fisher-KKP dynamics caused by chemotactic effects, and prove that

$$\sup_{0<\chi\leq\chi_1} \sup_{t_0\in\mathbb{R}, t\geq 0} \frac{1}{\chi} \|u_\chi(\cdot, t+t_0; t_0, u_0) - u_0(\cdot, t+t_0; t_0, u_0)\|_\infty < \infty$$

for every $0 < \chi_1 < \frac{b_{\inf}}{\mu}$ and every uniformly continuous initial function $u_0$, with $\inf_x u_0(x) > 0$, where $(u_\chi(x, t+t_0; t_0, u_0), v_\chi(x, t+t_0; t_0, u_0))$ denotes the unique

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classical solution of (0.1) with $u_\chi(x, t_0; t_0, u_0) = u_0(x)$, for every $0 \leq \chi < b_{\inf}$.


## 1. Introduction and statement of the main results

Chemotaxis, the ability for micro-organisms to respond to chemical signals by moving along the gradient of the chemical substance, plays important roles in a wide range of biological phenomena (see [4,20,23]), and accordingly a considerable literature is concerned with its mathematical analysis. We consider the following parabolic–elliptic chemotaxis system on $\mathbb{R}^N$ with space–time dependent logistic source,

$$\begin{cases} \partial_t u = \Delta u - \chi \nabla \cdot (u \nabla v) + u(a(x,t) - b(x,t)u), \quad x \in \mathbb{R}^N, \\ 0 = \Delta v - \lambda v + \mu u, \quad x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u(x, t)$ and $v(x, t)$ denote mobile species density and chemical density functions, respectively, $\chi$ is a positive constant which measures the sensitivity with respect to chemical signals, $a(x, t)$ and $b(x, t)$ are positive functions and measure the self growth and self limitation of the mobile species, respectively. The constant $\mu$ is positive and the term $+\mu u$ in the second equation of (1.1) indicates that the mobile species produce the chemical substance over time. The positive constant $\lambda$ measures the degradation rate of the chemical substance. System (1.1) is a space–time logistic source dependant variation of the celebrated parabolic–elliptic Keller–Segel chemotaxis systems (see [17,18]).

Note that (1.1) is a particular case of the following chemotaxis model,

$$\begin{cases} \partial_t u = \Delta u - \chi \nabla \cdot (u \nabla v) + u(a(x,t) - b(x,t)u), \quad x \in \Omega, \\ \tau v_t = \Delta v - \lambda v + \mu u, \quad x \in \Omega \end{cases} \tag{1.2}$$

complemented with certain boundary conditions if $\Omega \subset \mathbb{R}^N$ is a bounded domain, where $\tau \geq 0$ is a nonnegative constant link to the speed of diffusion of the chemical substance. Note that when $\tau = 0$ and $\Omega = \mathbb{R}^N$ in (1.2), we recover (1.1). Hence, (1.1) models the situation where the chemoattractant defuses very quickly and the underlying environment is very large.

It is well known that chemotaxis systems present very interesting dynamics. Indeed, when $\tau > 0$, $N = 2$, $a(x, t) \equiv b(x, t) \equiv 0$ and (1.2) is considered on a ball centered at origin associated with homogeneous Neumann condition, Herrero and Velàzquez [10] proved the existence of solutions which blow up at finite time. Under these assumptions but $\tau = 0$, Jäger and Lauckhaus [15] obtained similar results. Similar results were established by Nagai in [21]. We also refer the reader to [8,9,12,16,32–34] (and the references therein) for some other works on the finite-time blow up of solutions of (1.2). We refer the reader to [2] and the references therein for more insights in the studies of chemotaxis models.

When $a(x, t) > 0$ and $b(x, t) > 0$, it is known that the blow-up phenomena may be suppressed to some extent. Indeed, if $a(x, t)$ and $b(x, t)$ are positive constant functions, $\tau = 0$ and $\lambda = \mu = 1$, it is shown in [28] that if either $N \leq 2$ or $b > \frac{N-2}{N}\chi$, then for every nonnegative Hölder's continuous initial $u_0(x)$, (1.2) on bounded domain complemented with Neumann boundary condition possesses a unique bounded global classical solution $(u(x, t; u_0), v(x, t; u_0))$ with $u(x, 0; u_0) = u_0(x)$. Furthermore, if $b > 2\chi$, then the trivial steady state $(\frac{a}{b}, \frac{a}{b})$ is asymptotically stable with respect to nonnegative and non-identically zero perturbations. These results have been extended by the authors of the current paper, [24], to (1.1) on $\mathbb{R}^N$ when $a(x, t)$ and $b(x, t)$ are constant functions. The work [24] also studied some spreading properties of solutions to (1.1) with compactly supported initials. Recently, several studies have been concerned with establishing adequate conditions on the chemotaxis sensitivity $\chi$ and other parameters in (1.2) to ensure the existence of time global solutions and the stability of equilibria solutions. In this regard, we refer to