J. Math. Anal. Appl. 464 (2018) 911–923

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Fractal interpolation functions with partial self similarity ☆

Dah-Chin Luor

Department of Financial and Computational Mathematics, I-Shou University, No. 1, Sec. 1, Syuecheng Road., Dashu District, Kaohsiung City 84001, Taiwan

ARTICLE INFO

Article history:
Received 3 February 2016
Available online 23 April 2018
Submitted by B. Bongiorno

Keywords:
Fractals
Interpolation
Fractal interpolation functions
Fixed points
Self similarity

ABSTRACT

Let a data set $\Delta = \{(t_i, y_i) \in \mathbb{R} \times \mathbf{Y} : i = 0, 1, \cdots, N\}$ be given, where $t_0 < t_1 < t_2 < \cdots < t_N$ and $\mathbf{Y}$ is a complete metric space. In this article, fractal interpolation functions (FIFs) on $I = [t_0, t_N]$ corresponding to the set Δ are constructed by mappings $W_1, \cdots, W_N$. Each W_k is of the form $W_k = (L_k, M_k)$, where $L_k : J_k \to I_k$ is a homeomorphism and $M_k : J_k \times \mathbf{Y} \to \mathbf{Y}$ is continuous. Here $I_k = [t_{k-1}, t_k]$ and $J_k = [t_{j(k)}, t_{l(k)}]$, $j(k), l(k) \in \{0, 1, \cdots, N\}$, are subintervals of I which depend on k. In this construction, the length of J_k is not assumed to be larger than the length of I_k, and each L_k is not supposed to be a contraction. A FIF established by this method has a property of self similarity between its graph on J_k and on I_k. In this paper we give a construction of FIFs with locally self similar graphs. The stability and sensitivity of FIFs established in this way are also discussed.

1. Introduction

A fractal function is a function whose graph is the attractor of an iterated function system (IFS). A fractal interpolation function (FIF) is a continuous fractal function interpolating a given set of points. FIFs are the basis of a constructive approximation theory for nondifferentiable functions [5]. FIFs are also suitable to model experimental data with complicated structures in nature. For instance, the theory of FIFs can be applied to signal processing [23,24,39,40], and modeling coastlines and shapes [17–19]. See also the references given in these articles.

The concept of FIFs was first introduced by Barnsley [1,2] and has been developed by many researchers. Various types of FIFs have been constructed in different ways, including the hidden variable FIFs [2,7], the vector-valued FIFs [8,20], the bilinear FIFs [4], the spline FIFs [10,13], the Hermite FIFs [14,15,30], the rational FIFs [34,35], and the coalescence FIFs [11,12,32]. Many properties of FIFs, including box dimension, smoothness, calculus, stability, sensitivity, shape preservation, monotonicity preservation, convexity

☆ This work was supported by the Ministry of Science and Technology, Taipei, ROC, under Grant MOST 104-2115-M-214-004.
E-mail address: dclour@isu.edu.tw.

https://doi.org/10.1016/j.jmaa.2018.04.041
0022-247X/© 2018 Elsevier Inc. All rights reserved.

preservation, and approximation properties are also discussed in [1,3,4,12,16,25–29,31,33,36–38]. See also the books [2,21,22], and the references given in the literature.

Here we give a brief introduction to the construction of a FIF. The readers are referred to [2] for more details. Let $(\mathbf{Y}, d_{\mathbf{Y}})$ be a complete metric space. Consider the set of points $\Delta = \{(t_i, y_i) \in \mathbb{R} \times \mathbf{Y} : i = 0, 1, \cdots, N\}$, where $N > 1$ and $t_0 < t_1 < \cdots < t_N$. Suppose that all the data points in Δ are non-collinear. Let $\mathbf{X} = \mathbb{R} \times \mathbf{Y}$ and $\theta > 0$. Define a metric d on $\mathbf{X}$ by $d((t, y), (t^*, y^*)) = |t - t^*| + \theta d_{\mathbf{Y}}(y, y^*)$ for all points (t, y), (t^*, y^*) in $\mathbf{X}$. Then $(\mathbf{X}, d)$ is a complete metric space. For each $k = 1, \cdots, N$, define $L_k : \mathbb{R} \to \mathbb{R}$ by

$$L_k(t) = a_k t + b_k, \quad \text{where } a_k = \frac{t_k - t_{k-1}}{t_N - t_0},\ b_k = \frac{t_N t_{k-1} - t_0 t_k}{t_N - t_0}. \tag{1.1}$$

Note that $L_k([t_0, t_N]) = [t_{k-1}, t_k]$. Let $0 \le s < 1$, $c > 0$, and let $M_k : \mathbf{X} \to \mathbf{Y}$ be a function that satisfies

$$d_{\mathbf{Y}}(M_k(t, y), M_k(t^*, y)) \le c|t - t^*| \text{ for all } t, t^* \in \mathbb{R}, y \in \mathbf{Y}, \tag{1.2}$$

$$d_{\mathbf{Y}}(M_k(t, y), M_k(t, y^*)) \le s d_{\mathbf{Y}}(y, y^*) \text{ for all } t \in \mathbb{R}, y, y^* \in \mathbf{Y}, \tag{1.3}$$

$$M_k(t_0, y_0) = y_{k-1} \quad \text{and} \quad M_k(t_N, y_N) = y_k. \tag{1.4}$$

For each $k = 1, \cdots, N$, define $f_k : \mathbf{X} \to \mathbf{X}$ by

$$f_k(t, y) = (L_k(t), M_k(t, y)) \quad \text{for all } (t, y) \in \mathbf{X}. \tag{1.5}$$

If we choose $\theta = \frac{1-a}{2c}$, where $a = \max\{a_1, \cdots, a_N\}$, then by [2, Ch. VI, Theorem 4.1 and Theorem 4.2], each f_k is a contraction mapping on $\mathbf{X}$ and there is exactly one attractor $G \subseteq \mathbf{X}$ of the IFS $\{\mathbf{X}; f_1, \cdots, f_N\}$. The set G is the graph of a continuous function f which satisfies $f(t_k) = y_k$ for all $k = 0, \cdots, N$. This function f is called a fractal interpolation function corresponding to the set of points Δ.

In [1] Barnsley considered the case that $\mathbf{Y} = [a, b]$, $d_{\mathbf{Y}}(y, y^*) = |y - y^*|$, $\mathbf{X} = [t_0, t_N] \times [a, b]$, and $d((t, y), (t^*, y^*)) = \max\{|t - t^*|, |y - y^*|\}$ for $(t, y), (t^*, y^*) \in \mathbf{X}$. A FIF is constructed by the IFS $\{\mathbf{X}; f_1, \cdots, f_N\}$ in [1, Theorem 1] under the conditions that, for each $k = 1, \cdots, N$, $L_k : [t_0, t_N] \to [t_{k-1}, t_k]$ is a contractive homeomorphism such that $L_k(t_0) = t_{k-1}$, $L_k(t_N) = t_k$, and $M_k : \mathbf{X} \to [a, b]$ is a continuous function with (1.3)–(1.4) for some $0 \le s < 1$. In particular, if L_k is of the form (1.1) and $M_k(t, y) = c_k t + \alpha_k y + e_k$ for $k = 1, \cdots, N$, then each f_k is an affine map and α_k is called the vertical scaling factor in f_k. If $|\alpha_k| < 1$ for $k = 1, \cdots, N$, then each f_k is a contraction mapping on $\mathbf{X}$ and the obtained FIFs are called linear FIFs [1, Example 1]. In [1, Example 2] a class of M_k is considered, where $M_k(t, y) = \alpha_k y + q_k(t)$ and $q_k(t) = u(L_k(t)) - \alpha_k b(t)$. Here u and b are continuous functions such that $u(t_k) = y_k$ for all $k = 0, \cdots, N$ and $b(t_0) = y_0$, $b(t_N) = y_N$. In the case $b = L(u)$, where L is a bounded linear operator on the space of all continuous functions defined on $[t_0, t_N]$, the FIF associated with such a IFS is called α-fractal function associated with u and is denoted by u^α, where $\alpha = (\alpha_1, \cdots, \alpha_N)$ is a vector. The operator $\mathcal{F}^\alpha : u \mapsto u^\alpha$ is called an α-fractal operator. The developments of properties of $\mathcal{F}^\alpha$ delineated a theory which is referred to as fractal approximation theory. See [25–27,29,35]. To get FIFs with more flexibility in a general sense, Wang and Yu [38] considered a class of IFSs with variable vertical scaling parameters. Let $[t_0, t_N] = [0, 1]$ and $M_k(t, y) = \alpha_k(t) y + q_k(t)$ for each $k = 1, \cdots, N$. Here α_k and q_k are Lipschitz functions defined on $[0, 1]$ such that $\sup_{t \in [0,1]} |\alpha_k(t)| < 1$ and $q_k(0) = y_{k-1} - \alpha_k(0) y_0$, $q_k(1) = y_k - \alpha_k(1) y_N$. Moreover, in the case $q_k(t) = u(L_k(t)) - \alpha_k(t) b(t)$, where u is the piecewise linear interpolation function through the set of points Δ and b is the linear function through the points (t_0, y_0) and (t_N, y_N), the stability and sensitivity of FIFs were investigated.

In the literature mentioned above, various types of FIFs have been constructed by considering different forms of M_k, and L_k is usually assumed to be a contraction homeomorphism from $[t_0, t_N]$ to $[t_{k-1}, t_k]$ such that $L_k(t_0) = t_{k-1}$ and $L_k(t_N) = t_k$. In the article [6], a recurrent structure for IFSs is introduced and