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# Boundary Schwarz lemma for solutions to Poisson's equation

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A B S T R A C T

Let $\overline{\mathbb{D}}$ be the closure of the unit disk $\mathbb{D}$ in $\mathbb{C}$ and $g$ a continuous function of $\overline{\mathbb{D}}$. In this paper, we establish the Schwarz lemma at the boundary for solutions to the Poisson's equation: $\triangle f = g$.

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## 1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ be the unit disk, $\mathbb{T} = \{z \in \mathbb{C} : \ |z| = 1\}$ the unit circle, and $\overline{\mathbb{D}}$ the closure of $\mathbb{D}$, i.e., $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$. For $z \in \mathbb{D}$, the formal derivatives of a complex-valued function $f$ are defined by:

$$f_z = \frac{1}{2}\left(f_x - if_y\right) \ \text{ and } \ f_{\overline{z}} = \frac{1}{2}\left(f_x + if_y\right).$$

Then, $f$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if its Jacobian $J_f$ satisfies the following condition (cf. [9]): For every $z \in \mathbb{D}$,

$$J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 > 0.$$

Define the Laplacian of $f$ as follows:

$$\Delta f = 4 f_{z\bar{z}}.$$

The following equation is the so-called Poisson's equation:

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$$\Delta f = g, \tag{1.1}$$

where $f$ and $g \in C(\overline{\mathbb{D}})$, and $f \in C^2(\mathbb{D})$ maps $\overline{\mathbb{D}}$ into itself.

Obviously, when $g = 0$, any solution to the equation (1.1) is harmonic. Clearly, harmonic mappings are generalizations of analytic functions. See, e.g., [3] for more properties of harmonic mappings.

The classical Schwarz lemma states that any analytic function $f$ maps $\mathbb{D}$ into itself with $f(0) = 0$ satisfies $|f(z)| \leq |z|$ in $\mathbb{D}$.

It is well known that the Schwarz lemma has become a crucial theme in many branches of mathematical research for more than a hundred years. We refer the reader to [1,2,10–12,15,16] for generalizations and applications of this lemma.

The classical Schwarz lemma at the boundary is as follows:

**Theorem A.** ([4, Page 42]) *Suppose $f : \mathbb{D} \to \mathbb{D}$ is an analytic function with $f(0) = 0$, and, further, $f$ is analytic at $z = 1$ with $f(1) = 1$. Then, the following two conclusions hold:*

(1) $f'(1) \geq 1$.
(2) $f'(1) = 1$ *if and only if* $f(z) \equiv z$.

Theorem A has the following generalization.

**Theorem B.** ([10, Theorem 1.1′]) *Suppose $f : \mathbb{D} \to \mathbb{D}$ is an analytic function with $f(0) = 0$, and, further, $f$ is analytic at $z = \alpha \in \mathbb{T}$ with $f(\alpha) = \beta \in \mathbb{T}$. Then, the following two conclusions hold:*

(1) $\overline{\beta} f'(\alpha)\alpha \geq 1$.
(2) $\overline{\beta} f'(\alpha)\alpha = 1$ *if and only if* $f(z) \equiv e^{i\theta} z$, *where* $e^{i\theta} = \beta\alpha^{-1}$ *and* $\theta \in \mathbb{R}$.

We remark that, when $\alpha = \beta = 1$, Theorem B coincides with Theorem A.

The study on the boundary version of the Schwarz lemma has been attracted much attention. For more discussions in this line, see, e.g., [12] for functions with one complex variable, and [10,11] for functions with several complex variables.

As the first purpose of this paper, we establish the Schwarz lemma at the boundary for solutions to the equation (1.1). Our results are Theorem 1.1 and Theorem 1.2 below, which are generalizations of Theorem A and Theorem B, respectively.

In what follows, the notation $C^2(\mathbb{D}) \bigcap C(\mathbb{T})$ denotes the set of all functions which are second continuously differentiable in $\mathbb{D}$ and continuous on $\mathbb{T}$.

**Theorem 1.1.** *Suppose $f \in C^2(\mathbb{D}) \bigcap C(\mathbb{T})$ is a solution to the equation (1.1). If $f$ is differentiable at $z = 1$ with $f(0) = 0$ and $f(1) = 1$, then the following inequality holds:*

$$\mathrm{Re}\,[f_x(1)] \geq \frac{2}{\pi} - \frac{3}{4}\|g\|_\infty, \tag{1.2}$$

*where "$\mathrm{Re}\,[f_x(1)]$" means the real part of $f_x(1)$. Furthermore, the above inequality is sharp when $g = 0$.*

Here and in the rest of this paper, we assume that $\|g\|_\infty = \sup\limits_{z\in\mathbb{D}}\{|g(z)|\} < \frac{8}{3\pi}$.

**Remark 1.1.** It follows from Theorem A that if an analytic function $f$ in $\mathbb{D}$ satisfies the assumptions of the classical Schwarz lemma at the boundary, then $f'(1)$ is real. However, in the case of solutions to the Poisson's equation (1.1), the situation is different. This can be seen from the following example.

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