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# Integral estimates of conformal derivatives and spectral properties of the Neumann-Laplacian

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ABSTRACT

In this paper we study integral estimates of derivatives of conformal mappings $\varphi : \mathbb{D} \to \Omega$ of the unit disc $\mathbb{D} \subset \mathbb{C}$ onto bounded domains $\Omega$ that satisfy the Ahlfors condition. These integral estimates lead to estimates of constants in Sobolev–Poincaré inequalities, and by the Rayleigh quotient we obtain spectral estimates of the Neumann–Laplace operator in non-Lipschitz domains (quasidiscs) in terms of the (quasi)conformal geometry of the domains. Specifically, the lower estimates of the first non-trivial eigenvalues of the Neumann–Laplace operator in some fractal type domains (snowflakes) were obtained.



## 1. Introduction

### 1.1. *Estimates of conformal derivatives*

In the work [27] we obtained lower estimates of the first non-trivial eigenvalues of the Neumann–Laplace operator in the terms of integrals of complex derivatives (i.e. hyperbolic metrics) of conformal mappings $\varphi : \mathbb{D} \to \Omega$. Let us recall that the classical Koebe distortion theorem [10] gives the following estimates of the complex derivatives in the case of univalent analytic functions (conformal homeomorphisms): $\varphi : \mathbb{D} \to \Omega$ normalized so that $\varphi(0) = 0$ and $\varphi'(0) = 1$:

$$\frac{1-|z|}{(1+|z|)^3} \leq |\varphi'(z)| \leq \frac{1+|z|}{(1-|z|)^3}.$$

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The example of the Koebe function

$$\varphi(z) = \frac{z}{(1-z)^2}$$

which maps the unit disc $\mathbb{D}$ onto $\Omega = \mathbb{C} \setminus (-\infty, -1/4]$ shows that these estimates don't give even square integrability of the complex derivatives in arbitrary simply connected planar domains. But if $\Omega \subset \mathbb{C}$ is a simply connected planar domain of finite measure then by the simple calculation

$$\iint\limits_{\mathbb{D}} |\varphi'(z)|^2 \, dxdy = \iint\limits_{\mathbb{D}} J(z,\varphi) \, dxdy = |\Omega| < \infty.$$

(We identify the complex plane $\mathbb{C}$ and the real plane $\mathbb{R}^2$: $\mathbb{C} \ni z = x + iy = (x, y) \in \mathbb{R}^2$.)

Hence in special classes of domains we have better integral estimates of the complex derivatives than by the Koebe distortion theorem.

In the present work we study integral estimates of the complex derivatives in domains bounded by Jordan curves that satisfy the Ahlfors three point condition [2]. For this study we introduce a notion of hyperbolic (integral) $\alpha$-dilatation of $\Omega$:

$$Q(\alpha, \Omega) := \iint\limits_{\mathbb{D}} |\varphi'(z)|^\alpha \, dxdy = \iint\limits_{\Omega} \left| \left(\varphi^{-1}\right)'(w) \right|^{2-\alpha} dudv.$$

The finiteness of the hyperbolic $\alpha$-dilatation and its convergence hyperbolic interval

$$\mathrm{HI}(\Omega) := \{\alpha \in \mathbb{R} : Q(\alpha, \Omega) < \infty\}$$

doesn't depend on choice of a conformal mapping $\varphi : \mathbb{D} \to \Omega$ and can be reformulated in terms of the hyperbolic metrics [14]. Namely

$$\iint\limits_{\mathbb{D}} |\varphi'(z)|^\alpha \, dxdy = \iint\limits_{\mathbb{D}} \left( \frac{\lambda_{\mathbb{D}}(z)}{\lambda_{\Omega}(\varphi(z))} \right)^\alpha dxdy$$
$$= \iint\limits_{\Omega} \left| \left(\varphi^{-1}\right)'(w) \right|^{2-\alpha} dudv = \iint\limits_{\Omega} \left( \frac{\lambda_{\mathbb{D}}\left(\varphi^{-1}(w)\right)}{\lambda_{\Omega}(w)} \right)^{2-\alpha} dudv$$

where $\lambda_{\mathbb{D}}$ and $\lambda_{\Omega}$ are hyperbolic metrics in $\mathbb{D}$ and $\Omega$ [9]. Let us recall that the hyperbolic metrics generated by $\lambda_{\mathbb{D}}\left(\varphi^{-1}(w)\right)$ are equivalent for different choice of conformal homeomorphism $\varphi : \Omega \to \mathbb{D}$ because any other conformal homeomorphism $\psi^{-1} : \Omega \to \mathbb{D}$ is a composition of $\varphi^{-1}$ and a Möbius homeomorphism (that is an isometry of the hyperbolic metric).

**Remark 1.1.** For any bounded simply connected domain $(-1.78, 2] \subset \mathrm{HI}(\alpha, \Omega)$ [26,28]. A more detailed discussion about the hyperbolic $\alpha$-dilatation and its convergence hyperbolic interval can be found in Appendix.

In [26] we proved that if a number $\alpha > 2$ belongs to $\mathrm{HI}(\Omega)$ then $\Omega$ has finite geodesic diameter. By this reason we call domains that satisfy to a property $2 < \alpha \in \mathrm{HI}(\Omega)$ as conformal $\alpha$-regular domains [14].

In this paper we obtain estimates of $Q(\alpha, \Omega)$ for quasidiscs, that represent a large subclass of conformal $\alpha$-regular domains, with the help of the exact inverse Hölder inequality for Jacobians of quasiconformal mappings. Using the estimates for $Q(\alpha, \Omega)$ we obtain estimates of constants for Sobolev–Poincaré inequalities and as a result we obtain lower estimates for first non-trivial eigenvalues of the Laplace operator with the Neumann boundary condition.

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