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The use of two-point Taylor expansions in singular one-dimensional boundary value problems I

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ABSTRACT

We consider the second-order linear differential equation $(x+1)y''+f(x)y'+g(x)y = h(x)$ in the interval $(-1,1)$ with initial conditions or boundary conditions (Dirichlet, Neumann or mixed Dirichlet–Neumann). The functions $f(x)$, $g(x)$ and $h(x)$ are analytic in a Cassini disk $\mathcal{D}_r$ with foci at $x = \pm 1$ containing the interval $[-1,1]$. Then, the end point of the interval $x = -1$ may be a regular singular point of the differential equation. The two-point Taylor expansion of the solution $y(x)$ at the end points ± 1 is used to study the space of analytic solutions in $\mathcal{D}_r$ of the differential equation, and to give a criterion for the existence and uniqueness of analytic solutions of the boundary value problem. This method is constructive and provides the two-point Taylor approximation of the analytic solutions when they exist.

1. Introduction

In [6] we considered the second-order linear equation $y'' + f(x)y' + g(x)y = h(x)$ in the interval $(-1,1)$ with initial conditions or boundary conditions of the type Dirichlet, Neumann or mixed Dirichlet–Neumann. The functions $f(x)$, $g(x)$ and $h(x)$ are analytic in a Cassini disk with foci at $x = \pm 1$ containing the interval $[-1,1]$. Then, the end points of the interval, where the boundary data are given, are regular points of the differential equation. The two-point Taylor expansion of the solution $y(x)$ at the end points ± 1 was used to give a criterion for the existence and uniqueness of analytic solutions of the initial or boundary value problem and approximate the solutions when they exist.

In this paper we continue our investigation considering problems that have an extra difficulty: one of the end points of the interval is a regular singular point of the differential equation. These problems are of great interest, since many special functions of the mathematical physics satisfy this type of equations [8]. We consider initial or boundary value problems of the form:

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$$\begin{cases} (x+1)y'' + f(x)y' + g(x)y = h(x) \quad \text{in} \quad (-1,1), \\ B \begin{pmatrix} y(-1) \\ y(1) \\ y'(-1) \\ y'(1) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \end{cases} \tag{1}$$

where f, g and h are analytic in a Cassini disk with foci at $x = \pm 1$ containing the interval $[-1, 1]$ (we give more details in the next section), $\alpha, \beta \in \mathbb{C}$ and B is a 2×4 matrix of rank two which defines the initial conditions or the boundary conditions (Dirichlet, Neumann or mixed).

The consideration of the interval $(-1, 1)$ is not a restriction, as any real interval (a, b) can be transformed into the interval $(-1, 1)$ by means of an affine change of the independent variable. The form of the differential equation in (1) is not a restriction either: consider a differential equation of the form $(x+1)^2u''(x) + (x+1)F(x)u'(x) + G(x)u(x) = 0$, with F and G analytic at $x = -1$. After the change of the dependent variable $u = (x+1)^\lambda y$, with λ a solution of the equation $\lambda(\lambda-1)+F(-1)\lambda+G(-1) = 0$, the equation may be written in the form $(x + 1)y'' + f(x)y' + g(x)y = 0$, with f and g analytic at $x = -1$. On the other hand, the point $x = -1$ is indeed a regular singular point of the differential equation when $|f(-1)| + |g(-1)| + |h(-1)| \neq 0$; if $f(-1) = g(-1) = h(-1) = 0$, then $x = -1$ is a regular point, and problem (1) is the regular problem analyzed in [6]. We omit this restriction here and then, the regular case studied in [6] may be considered a particular case of the more general one analyzed in this paper.

A standard theorem for the existence and uniqueness of solution of (1) is based on the knowledge of the two-dimensional linear space of solutions of the homogeneous equation $(x + 1)y'' + f(x)y' + g(x)y = 0$ [2, Chap. 4, Sec. 1]. When f are g are constants or in some other particular situation, it is possible to find the general solution of the equation (sometimes via the Green function [2, Chap. 4], [10, Chaps. 1 and 3])). But this is not possible in general, and that standard criterion for the existence and uniqueness of solution of (1) is not practical. Other well-known criterion for the existence and uniqueness of solution of (1) is based on the Lax–Milgram theorem when (1) is an elliptic problem [3]. In any case, the determination of the existence and uniqueness of solution of (1) requires a non-systematic detailed study of the problem, like for example the study of the eigenvalue problem associated to (1) [2, Chap. 4, Sec. 2], [10, Chap. 7].

When f, g and h are analytic in a disk with center at $x = 0$ and containing the interval $[-1, 1]$, we may consider the initial value problem:

$$\begin{cases} (x+1)y'' + f(x)y' + g(x)y = h(x), \quad x \in (-1,1), \\ y(0) = y_0, \quad y'(0) = y_0', \end{cases} \tag{2}$$

with $y_0, y_0' \in \mathbb{C}$. Using the Frobenius method we can approximate the solution of this problem by its Taylor polynomial of degree $N \in \mathbb{N}$ at $x = 0$, $y_N(x) = \sum_{n=0}^{N} c_k x^k$, where the coefficients c_k are affine functions of $c_0 = y_0$ and $c_1 = y_0'$. By imposing the boundary conditions given in (1) over $y_N(x)$, we obtain an algebraic linear system for y_0 and y_0'. The existence and uniqueness of solution of this algebraic linear system gives us information about the existence and uniqueness of solution of (1). This procedure, although theoretically possible, has a difficult practical implementation since the data of the problem are given at $x = \pm 1$, not at $x = 0$ (see [1,9]). Moreover, when f, g or h have a singularity close to the interval $[-1, 1]$, the above mentioned disk does not contain the interval $[-1, 1]$ and the Taylor series of the solution $y(x)$ does not converge $\forall x \in [-1, 1]$. In this case we can use a Taylor expansion of the solution at several points along the interval $[-1, 1]$ and match these expansions at intersecting disks [7, Sec. 7]. In this way, we obtain an approximation of the solution of (1) in the form of a piecewise polynomial in several subintervals of $[-1, 1]$. But this approximation is not uniform in the whole interval $[-1, 1]$ and the matching of the expansions translates into numerical errors.

In [6] we improved the ideas of the previous paragraph for the regular case (when $f(-1) = g(-1) = h(-1) = 0$) using, not the standard Taylor expansion in the associated initial value problem (2), but a

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