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A lattice sum involving the cosine function

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ABSTRACT

In this article we prove that, as $n \to \infty$,

$$\sum_{j,k=1}^{n-1} \frac{1}{3 - \cos\frac{2\pi j}{n} - \cos\frac{2\pi k}{n} - \cos\frac{2\pi(j+k)}{n}} \sim \frac{n^2 \log n}{\sqrt{3}\pi}.$$

We also obtain the secondary term of size $\asymp n^2$ to be followed by an error term of size $O(\log n)$. In this work, results and techniques from classical analysis involving roles by several special functions, from number theory, and from numerical analysis are needed.

1. Introduction

The sum

$$S_n := \sum_{j,k=1}^{n-1} \frac{1}{3 - \cos\frac{2\pi j}{n} - \cos\frac{2\pi k}{n} - \cos\frac{2\pi(j+k)}{n}} \quad (n \geq 2) \tag{1.1}$$

$$= \frac{1}{2} \sum_{j,k=1}^{n-1} \frac{1}{\sin^2\frac{\pi j}{n} + \sin^2\frac{\pi k}{n} + \sin^2\frac{\pi(j+k)}{n}} \tag{1.2}$$

and other sums and integrals of similar types have been significant in studies of lattice Green's functions which arise in many problems in condensed matter physics, random walks on lattices and the calculation of the resistance of resistor networks (see e.g. [7] and [16]), and in studies of metrized graphs [6]. Since even the order of magnitude of S_n as $n \to \infty$ wasn't known, in this paper we study this sum.

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It is not difficult to obtain the non-trivial lower bound

$$S_n > \frac{n^2 \log n}{4\pi^2} + O(n^2) \tag{1.3}$$

by using

$$\sin^2 \theta \le \theta^2, \quad \forall \theta \in \mathbb{R} \tag{1.4}$$

in (1.2). To get an upper bound one can employ a result due to Montgomery [11]:

$$(\sin \pi\theta)^{-2} \le (\pi||\theta||)^{-2} + c, \; ||\theta|| := \min(\theta - \lfloor\theta\rfloor, \lceil\theta\rceil - \theta), \; c := 1 - \frac{4}{\pi^2}, \quad \forall \theta \in \mathbb{R}. \tag{1.5}$$

Upon plugging (1.5) in (1.2), in all of its occurrences the constant c may be neglected if we are just trying to determine the order of magnitude of S_n. This leads to

$$S_n \ll n^2 \log n, \tag{1.6}$$

which in conjunction with (1.3) gives the order of magnitude of S_n:

$$S_n \asymp n^2 \log n. \tag{1.7}$$

Our main result is

Theorem. *As $n \to \infty$,*

$$\begin{aligned}\sum_{j,k=1}^{n-1} &\frac{1}{3 - \cos\frac{2\pi j}{n} - \cos\frac{2\pi k}{n} - \cos\frac{2\pi(j+k)}{n}} \\ &= \frac{n^2 \log n}{\sqrt{3}\pi} + \frac{n^2}{\sqrt{3}\pi}\Big(\gamma - \frac{\sqrt{3}\pi}{6} + \log(4\pi\sqrt[4]{3}) - 3\log\Gamma\big(\frac{1}{3}\big)\Big) + O\big(\log n\big),\end{aligned} \tag{1.8}$$

where γ is Euler's constant.

The following two sections contain two proofs for the asymptotic value of S_n. The rest of the paper is for obtaining the secondary term and the error term. We achieve this by first relating S_n to an integral by the methods of numerical analysis (which is the main cause of the error term $O(\log n)$ in (1.8)), and then carrying out the evaluations of the terms which appear in the relation. It is desirable to obtain a more precise version of (1.8). In fact, since the summands are algebraic numbers in the cyclotomic field obtained by adjoining a primitive n-th root of unity to $\mathbb{Q}$, due to the symmetry involved in the sum the precise value of S_n must be a rational number ($S_2 = \frac{1}{4}$, $S_3 = \frac{10}{9}$, $S_4 = \frac{44}{16}$, $S_5 = \frac{58}{11}$, $S_6 = \frac{1577}{180}, \dots$) We wonder whether a method for determining this rational number for any given n can be developed.

2. Proof of the asymptotic value

We first observe that by applying changes of variables the summation variables in (1.2) may be brought into the domain $0 < |j|, |k| \le \frac{n}{2}$ without changing the summands. (For example, for the portion $0 < j \le \frac{n}{2}$, $\frac{n}{2} < k < n$, $\frac{n}{2} < j + k \le n$ we take $k - n$ as our new k and leave j unchanged.) This will allow us to use Taylor expansion around the origin. First we deal with those j, k pairs which are not close to the origin. If $j^2 + k^2 \ge \frac{n^2}{72}$, say, then $\max(\frac{|j|}{n}, \frac{|k|}{n}) \ge \frac{1}{12}$. Hence each such summand will satisfy

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