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On the Rayleigh–Taylor instability in compressible viscoelastic fluids

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A R T I C L E I N F O

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A B S T R A C T

We mathematically investigate the Rayleigh–Taylor (abbr. RT) instability in the compressible viscoelastic fluid in the presence of a uniform gravitational field in a bounded domain based on Oldroyd-B model. We first analyze the linearized equations around the viscoelastic RT equilibrium solution, and obtain an instability condition. Then we construct solutions of the linearized viscoelastic RT problem that grow in time in the Sobolev space H^3 under an instability condition, thus leading to the linear instability. Finally, with the help of the constructed unstable solutions of the linearized viscoelastic RT problem and a local well-posedness result of smooth solutions to the nonlinear viscoelastic RT problem, we mathematically prove the instability of viscoelastic RT (abbr. VRT) problem in the sense of Hadamard.

1. Introduction

The equilibrium of the heavier fluid on top of the lighter one under the gravity is unstable, and such instability is called the Rayleigh–Taylor (abbr. RT) instability. In this case, the equilibrium state is unstable to sustain small disturbances, and this unstable disturbance will grow and lead to a release of potential energy, as the heavier fluid moves down under the gravitational force, and the lighter one is displaced upwards. This phenomenon was first studied by Rayleigh [28] and then Taylor [29], and is called therefore the RT instability. The mathematical proof of RT instability have been extensively established in the sense of Hadamard, see [14,15,20,21]. It has been also widely investigated how the RT instability evolves under the effects of other physical factors, such as rotation [3], internal surface tension [6,30,32], magnetic fields [2,16,18,19,22,23,31] and so on.

Recently, the RT instability in the incompressible viscoelastic fluid have been investigated, see [13,24,25]. In this article, we further mathematically prove the RT instability in compressible viscoelastic fluid based on the following idea Oldroyd-B model in the presence of a uniform gravitational field:

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$$\begin{cases} \rho_t + \mathrm{div}(\rho v) = 0, \\ \rho v_t + \rho v \cdot \nabla v + \nabla P - \mu_1 \Delta v - \mu_2 \nabla \mathrm{div} v = \kappa \mathrm{div}\,(\rho U U^{\mathrm{T}}) - \rho g e_3, \\ U_t + v \cdot \nabla U - \nabla v U = 0. \end{cases} \tag{1.1}$$

Here the unknowns $\rho := \rho(t, x)$, $v := v(t, x)$, and $U := U(t, x)$ denote the density, velocity, and deformation tensor (a 3×3 matrix valued function), respectively. $\kappa > 0$, and $g > 0$ stand for the elastic coefficient and gravitational constant, respectively. $e_3 := (0, 0, 1)^{\mathrm{T}}$ is the vertical unit vector, and $-\rho g e_3$ represents the gravitational force. $\mu_1 > 0$ is the coefficient of shear viscosity and $\mu_2 := \nu + \mu_1/3$ with ν being the positive bulk viscosity. In this article, the pressure function $P := P(\rho)$ is always assumed to be smooth, positive, and strictly increasing with respect to the density ρ. In the system (1.1), the equation $(1.1)_1$ is a continuity equation, $(1.1)_2$ describes the balance law of momentum, while $(1.1)_3$ is called the deformation equation. The well-posedness problem of the equations (1.1) without the gravity has been widely investigated by many authors, see [9–12] for examples.

To investigate the RT instability of the above equations, we shall construct an equilibrium state to the equations (1.1). To begin with, we choose a density profile $\bar{\rho} := \bar{\rho}(x_3)$, which is independent of (x_1, x_2) and satisfies

$$\bar{\rho} \in C^4(\bar{\Omega}), \quad \inf_{x \in \Omega} \bar{\rho} > 0, \tag{1.2}$$

and the RT condition

$$\bar{\rho}'(x_3^0)|_{x_3 = x_3^0} > 0 \quad \text{for some } x_3^0 \in \{x_3 \mid (x_1, x_2, x_3)^T \in \Omega\}. \tag{1.3}$$

The RT condition assures that there is at least a region in which the RT density has larger density with increasing height x_3, thus leading to the classical RT instability [17].

Then, for given $\bar{\rho}$ and g, we define

$$\bar{U} := \begin{pmatrix} \bar{u} & 0 & 0 \\ 0 & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{pmatrix}$$

and

$$\bar{u} \equiv \bar{u}(x_3) := \pm \sqrt{\frac{F(P'(\bar{\rho})\bar{\rho}' + g\bar{\rho}) + C}{\kappa \bar{\rho}}}, \tag{1.4}$$

where $F(P'(\bar{\rho})\bar{\rho}' + g\bar{\rho})$ denotes a primitive function of $P'(\bar{\rho})\bar{\rho}' + g\bar{\rho}$ and C is a positive constant satisfying

$$\inf_{x \in \bar{\Omega}} \{F(P'(\bar{\rho})\bar{\rho}' + g\bar{\rho}) + C\} > 0.$$

It is easy to see that (1.4) makes sense for a bounded domain Ω, and

$$P'(\bar{\rho})\bar{\rho}' = \kappa(\bar{\rho}\bar{u}^2)' - g\bar{\rho} \tag{1.5}$$

where $P'(\bar{\rho}) = P'(s)|_{s = \bar{\rho}}$ and $\bar{\rho}'(x_3) := \mathrm{d}/\mathrm{d}x_3$. Thus, we immediately see that $(\bar{\rho}, 0, \bar{U})$ is an equilibrium solution of (1.1).

Now, we denote the perturbation around the equilibrium state by

$$\varrho = \rho - \bar{\rho}, \quad v = v - 0, \quad V = U - \bar{U}$$