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A note on matrices mapping a positive vector onto its element-wise inverse

Sébastien Labbé

CNRS, LaBRI, UMR 5800, F-33400 Talence, France

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ABSTRACT

For any primitive matrix $M \in \mathbb{R}^{n\times n}$ with positive diagonal entries, we prove the existence and uniqueness of a positive vector $\mathbf{x} = (x_1, \dots, x_n)^t$ such that $M\mathbf{x} = (\frac{1}{x_1}, \dots, \frac{1}{x_n})^t$. The contribution of this note is to provide an alternative proof of a result of Brualdi et al. (1966) [1] on the diagonal equivalence of a nonnegative matrix to a stochastic matrix.

1. Introduction

In this note, we consider matrices mapping a vector with positive entries onto its element-wise inverse. We prove unicity and existence of such a vector for primitive matrices, that is nonnegative matrices some power of which is positive, with positive diagonal entries. The main result is:

Theorem 1. *Let $M \in \mathbb{R}^{n\times n}_{\geq 0}$ be a primitive matrix with positive diagonal entries. Then there exists a unique vector $\mathbf{x} = (x_1, \dots, x_n)^t$ with positive entries such that $M\mathbf{x} = (\frac{1}{x_1}, \dots, \frac{1}{x_n})^t$.*

It turns out that this question was already answered in 1966 under an equivalent form. In [1], it was proved that if A is a nonnegative square matrix with positive diagonal entries, then there exists a unique diagonal matrix D with positive diagonal entries such that DAD is row stochastic (see also [3] who proved it for positive matrices A). The equivalence is explained below in Lemma 2.

As a consequence, the contribution of this note is to provide an alternative proof of the above result. Unicity for primitive matrices is obtained as a consequence of Perron theorem whereas existence for nonnegative matrices with positive diagonal entries is deduced from the Brouwer fixed-point theorem.

E-mail address: sebastien.labbe@labri.fr.
URL: http://www.slabbe.org/.

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This note is structured as follows. In Section 2, we present an equivalent system of quadratic equations to be solved. In Section 3, we deduce unicity for primitive matrices from Perron theorem. In Section 4, we reduce the question to finding fixed-points of a function and we recall Brouwer and Banach fixed-point theorems in Section 5. In Section 6, we use Brouwer fixed-point theorem to prove existence for nonnegative matrices with positive diagonal entries. In Section 7, we use Banach fixed-point theorem to prove existence and unicity for nonnegative matrices with relatively large enough diagonal entries including matrices which are not primitive (Proposition 12).

2. A system of quadratic equations

We say that a vector or a matrix is *nonnegative* (resp. *positive*) if all of its entries are nonnegative (resp. positive).

Lemma 2. *Let $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ be a nonnegative matrix and $\mathbf{x} = (x_1, \dots, x_n)^t$ be a positive vector. The following conditions are equivalent.*

(i) $M\mathbf{x} = (\frac{1}{x_1}, \dots, \frac{1}{x_n})^t$,
(ii) $\mathrm{diag}(\mathbf{x}) M \mathrm{diag}(\mathbf{x})$ *is a stochastic matrix,*
(iii) *for every $i \in \{1, \dots, n\}$,*

$$x_i \sum_{j=1}^{n} m_{ij} x_j = 1. \tag{1}$$

Proof. (i) $\iff$ (ii). The matrix $\mathrm{diag}(\mathbf{x}) M \mathrm{diag}(\mathbf{x})$ is stochastic if and only if $(1, \dots, 1)^t$ is a right eigenvector with eigenvalue 1, that is,

$$\mathrm{diag}(\mathbf{x}) M \mathrm{diag}(\mathbf{x}) (1, \dots, 1)^t = (1, \dots, 1)^t \tag{2}$$

which is equivalent to $M\mathbf{x} = \mathrm{diag}(\mathbf{x})^{-1}(1, \dots, 1)^t = (\frac{1}{x_1}, \dots, \frac{1}{x_n})^t$.

(ii) $\iff$ (iii). Let $\mathbf{r}_i$ be the i-th row of the matrix M. We develop (2) and we get

$$\mathrm{diag}(\mathbf{x}) M\mathbf{x} = \mathrm{diag}(\mathbf{x})(\mathbf{r}_1 \cdot \mathbf{x}, \dots, \mathbf{r}_n \cdot \mathbf{x})^t = (x_1 \mathbf{r}_1 \cdot \mathbf{x}, \dots, x_n \mathbf{r}_n \cdot \mathbf{x})^t = (1, \dots, 1)^t.$$

This equation is verified if and only if, for each $i \in \{1, \dots, n\}$, the quadratic Equation (1) in $x_1, \dots, x_n$ holds. □

The system of equations (1) for $i \in \{1, \dots, n\}$ is illustrated in Fig. 1 for $n = 2$ and $n = 3$.

3. Uniqueness for primitive matrices

A *primitive* matrix is a nonnegative matrix some power of which is positive.

Lemma 3. *Let $M \in \mathbb{R}^{n\times n}_{\geq 0}$ be a primitive matrix and $v \in \mathbb{R}^n_{>0}$. If $M^2 v = v$, then $Mv = v$.*

Proof. We already have that v is a positive vector fixed by M^2 which is primitive. But so is Mv:

$$M^2(Mv) = M(M^2 v) = Mv.$$

By Perron's theorem, v and Mv must be colinear, that is, there exists $\lambda \in \mathbb{R}$ such that $v = \lambda M v$. Then, $v = \lambda^2 M^2 v = \lambda^2 v$ and thus $\lambda^2 = 1$. Since v and Mv are positive, we deduce $\lambda = 1$. □

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