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Sensitive dependence for nonautonomous discrete dynamical systems ☆

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A B S T R A C T

Given a nonautonomous discrete dynamical system (NDS) $(X, f_{1,\infty})$ we show that transitivity and density of periodic points do not imply sensitivity in general, i.e., in the definition of Devaney chaos there are no redundant conditions for NDS. In addition, we show that if we also assume uniform convergence of the sequence (f_n) that induces the NDS, then sensitivity follows. Furthermore, in contrast to the autonomous case, we show that there exist minimal NDS which are neither equicontinuous nor sensitive.

1. Introduction

Nonautonomous discrete dynamical systems (NDS for short) were introduced by S. Kolyada and L. Snoha in [11]. A very readable account of some recent developments on the theory of nonautonomous discrete dynamical systems has been given in [3].

Let (X, d) a metric space. An NDS is a pair $(X, f_{1,\infty})$ where $f_{1,\infty} = (f_n : X \to X)_{n\in\mathbb{N}}$ is a sequence of continuous functions. The composition

$$f_1^n := f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$$

is said to be the nth-iterate of $f_{1,\infty}$ for all $n \in \mathbb{N}$ and, as usual, the symbol f_1^0 will stand for Id_X, the identity map on X. The *orbit* of a point $x \in X$ is the set

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$$\mathcal{O}_{f_{1,\infty}}(x) := \{x, f_1^1(x), f_1^2(x), \cdots f_1^n(x), \cdots\}$$

which can also be described by the difference equation $x_1 = x$ and $x_{n+1} = f_n(x_n)$ for each $n \in \mathbb{N}$.

Recall that given a compact metric space (X, d) and a sequence of continuous functions $(f_n : X \to X)_{n\in\mathbb{N}}$, a general form of a nonautonomous difference equation is the following:

$$\begin{cases} x_0 = x, \\ x_{n+1} = f_n(x_n) \end{cases}$$

for each $x \in X$. This kind of nonautonomous difference equation has been considered by several mathematicians (see for instance [17], [18] among others). Classical examples deal with $X = [0, 1]$ the unit interval endowed with the usual metric.

NDS generalize autonomous discrete systems (ADS for short). An ADS is given by (X, f), where (X, d) is a metric space and $f : X \to X$ is continuous. These systems can be seen as a particular case of NDS just by considering $f_n = f$ for all $n \in \mathbb{N}$.

In this paper we address some questions related to transitivity and sensitivity on NDS. Our motivation is Devaney's definition of chaos [7]. Devaney defined an ADS (X, f) to be chaotic if it satisfies the following three conditions: (i) (X, f) is transitive, (ii) the set of periodic points of (X, f) is a dense subset of X and (iii) (X, f) is sensitive. Banks et al. [5] proved that conditions (i) and (ii) imply condition (iii) (to avoid degenerate cases X is assumed to be infinite).

These results suggest the question whether the two former conditions, i.e. transitivity and density of periodic points, in an NDS imply the last one, i.e. sensitivity. This was proposed by Lan [14] in full generality:

In nonautonomous dynamical systems, does transitivity together with density of periodic points imply sensitivity?

Under additional conditions, considering the density of k-periodic points of X, Zhu et al. [18, Theorems 3.1 and 3.2] answered this question positively. However, in [16, Example 4.4] the authors provided a negative answer for an NDS on the interval $[0, 1]$. In the papers by Zhu et al. [18], Lan [14] and Sanchez et al. [16] the authors considered a periodic point for the NDS to be $x \in X$ satisfying $f_1^n(x) = x$ for some $n \in \mathbb{N}$. In our paper we will consider periodic points to be those $x \in X$ satisfying that $f_1^{nk}(x) = x$ for some $n \in \mathbb{N}$ and any $k \in \mathbb{N}$. This definition coincides with the classical one of periodic point in the ADS case.

The paper is organized as follows. Section 2 is devoted to Devaney's chaos for NDS. We will provide new examples which give a negative answer to the question proposed by Lan [14]. Indeed, for each transitive non-sensitive ADS we show the existence of an NDS $(X, f_{1,\infty})$ which is transitive, has a dense subset of periodic points but fails to be sensitive.

The main result of this section gives a positive answer to Lan's question and states that if (X, d) is a metric space without isolated points and $(X, f_{1,\infty})$ is an NDS such that $f_n : X \to X$ converges to f uniformly, then transitivity of the NDS and density of periodic points in X imply sensitivity.

In Section 3 we deal with equicontinuity and sensitivity. We show that if a transitive NDS $(X, f_{1,\infty})$ is equicontinuous at x_0, then x_0 is a transitive point. As a consequence, if an NDS $(X, f_{1,\infty})$ is topologically transitive and equicontinuous, then it is minimal. We finish with an example of a minimal NDS which is neither equicontinuous nor sensitive. This fact is a significant difference between the theory of ADS and the theory of NDS. Further references in ADS can be found in [9]. In [2,3] the reader can find further information about NDS. The interested reader in chaos, transitivity and sensitivity for NDS might consult [4,8,10,15].

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