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The unit ball of an injective operator space has an extreme point

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ABSTRACT

We define an AW^*-TRO as an off-diagonal corner of an AW^*-algebra, and show that the unit ball of an AW^*-TRO has an extreme point. In particular, the unit ball of an injective operator space has an extreme point, which answers a question raised in [8] affirmatively. We also show that an AW^*-TRO (respectively, an injective operator space) has an ideal decomposition, that is, it can be decomposed into the direct sum of a left ideal, a right ideal, and a two-sided ideal in an AW^*-algebra (respectively, an injective C^*-algebra). In particular, we observe that an AW^*-TRO, hence an injective operator space, has an algebrization which admits a quasi-identity.

Recall that an operator space X is called a *triple system* or a *ternary ring of operators* (*TRO* for short) if there exists a complete isometry ι from X into a C^*-algebra such that $\iota(x)\iota(y)^*\iota(z) \in \iota(X)$ for all $x, y, z \in X$. A theorem of Ruan and Hamana (independently) states that an operator space X is injective if and only if it is an off-diagonal corner of an injective C^*-algebra, i.e., there exist an injective C^*-algebra $\mathcal{A}$ and projections $p, q \in \mathcal{A}$ (meaning $p = p^2 = p^*$ and $q = q^2 = q^*$) such that X is completely isometric to $p\mathcal{A}q$ (Theorem 4.5 in [14] and Theorem 3.2 (i) in [2]). In particular, an injective operator space is a TRO. Noting that an injective C^*-algebra is monotone complete and hence an AW^*-algebra, the Ruan–Hamana theorem motivates the following definition. (The reader is referred to [15] for a modern account of and recent progress in monotone complete C^*-algebras and AW^*-algebras.)

Definition 1. We say that an operator space X is an AW^***-TRO** if there exist an AW^*-algebra $\mathcal{A}$ and projections $p, q \in \mathcal{A}$ such that X is completely isometric to $p\mathcal{A}q$.

Remark 2.

(1) While a different definition of an AW^*-TRO was given in [12] (Definition 6.2.1), where the definition is that the linking C^*-algebra is AW^*, we think our definition is the right one. For instance, a countably-

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infinite-dimensional column Hilbert space is an injective operator space ([13]) and hence a TRO, but its linking C^*-algebra is not unital (and hence not AW^*), while the space is an AW^*-TRO using our definition. Note also that a W^*-TRO can have a linking C^*-algebra which is not a W^*-algebra.

(2) With our definition of an AW^*-TRO, all Theorems, Corollaries, and Lemmas (excluding the second assertion of Lemma 6.2.9) in Section 6.2 of [12] remain to be valid replacing TT^* by $p\mathcal{A}p$, T^*T by $q\mathcal{A}q$, and $\mathscr{L}_T = \begin{pmatrix} TT^* & T \\ T^* & T^*T \end{pmatrix}$ by $\mathscr{L}_T = \begin{pmatrix} p\mathcal{A}p & p\mathcal{A}q \\ q\mathcal{A}p & q\mathcal{A}q \end{pmatrix}$ $(\subseteq M_2(\mathcal{A}))$, where $\mathcal{A}$ is an AW^*-algebra, and $p, q \in \mathcal{A}$ are projections, and T is an AW^*-TRO identified with $p\mathcal{A}q$. In addition, instead of Definition 6.3.1 in [12], if we define T to be of *type I*, *type II*, or *type III* if $\mathcal{A}$ can be chosen to be of type I, type II, or type III, respectively, then all Theorems and Lemmas (excluding item 3 of Lemma 6.3.5) in Section 6.3 of [12] also remain to be valid.

Theorem 3. *The unit ball (always assumed to be norm-closed) of an AW^*-TRO has an extreme point. In particular, the unit ball of an injective operator space has an extreme point, which answers a question raised in [8] (Question 2) affirmatively.*

Proof. Let X be an AW^*-TRO. We may assume that $X = p\mathcal{A}q$, where $\mathcal{A}$ is an AW^*-algebra and $p, q \in \mathcal{A}$ are projections. By the comparison theorem in [3], there exist unique central projections $r, t, l \in \mathcal{A}$ satisfying $r + t + l = 1$ such that $rp \prec rq$, $tp \sim tq$, and $lp \succ lq$. (Here $rp \prec rq$ means $rp \preceq rq$ but $rp \nsim rq$, however, $0 \prec 0$ is allowed.) That is, there exist partial isometries $u, v, w \in \mathcal{A}$ such that $uu^* = rp$, $u^*u \leq rq$, $vv^* = tp$, $v^*v = tq$, $ww^* \leq lp$, and $w^*w = lq$. Let $e := u + v + w \,(\in p\mathcal{A}q)$, then it is easy to check that $(p - ee^*)\mathcal{A}(q - e^*e) = \{0\}$. Thus by a variation of Kadison's theorem (Theorem 1 in [4]; see Proposition 1.4.8 in [11] or Proposition 1.6.5 in [16] for the variation we need here), e is an extreme point of the unit ball of $p\mathcal{A}q$. □

From the proof above we obtain "ideal decompositions" for AW^*-TROs and injective operator spaces similar to the ones done for TROs with predual in [7]. The technique we use here is to embed an off-diagonal corner into the diagonal corners which is a modification of the technique developed in [1] and is employed in [7].

Corollary 4. *An AW^*-TRO (respectively, an injective operator space) can be decomposed into the direct sum of TROs X_T, X_L, and X_R:*

$$X = X_T \overset{\infty}{\oplus} X_L \overset{\infty}{\oplus} X_R$$

so that there is a complete isometry ι from X into an AW^*-algebra (respectively, an injective C^*-algebra) in which $\iota(X_T)$, $\iota(X_L)$, and $\iota(X_R)$ are a two-sided, left, and right ideal, respectively, and

$$\iota(X) = \iota(X_T) \overset{\infty}{\oplus} \iota(X_L) \overset{\infty}{\oplus} \iota(X_R).$$

Proof. Let X be an AW^*-TRO, and assume that $X = p\mathcal{A}q$, where $\mathcal{A}$ is an AW^*-algebra and $p, q \in \mathcal{A}$ are projections. Let $r, t, l \in p\mathcal{A}q$ as in the proof of Theorem 3, and put $X_T := tX$, $X_L := lX$, and $X_R := rX$, then $X = X_T \overset{\infty}{\oplus} X_L \overset{\infty}{\oplus} X_R$. Let $\mathcal{B} := p\mathcal{A}p \overset{\infty}{\oplus} q\mathcal{A}q$ which is an AW^*-algebra since $p\mathcal{A}p$ and $q\mathcal{A}q$ are so by Theorem 2.4 in [10]. For each $x \in X$, let $x_T := tx$, $x_L := lx$, and $x_R := rx$, and define a mapping $\iota : X \to \mathcal{B}$ by $\iota(x) := (x_T + x_L)e^* \oplus e^*x_R$, where e is as in the proof of Theorem 3. Then clearly $\iota(X) = \iota(X_T) \overset{\infty}{\oplus} \iota(X_L) \overset{\infty}{\oplus} \iota(X_R)$. We claim that ι is a complete isometry. $\|\iota(x)\| = \max\{\|(x_T + x_L)e^*\|, \|e^*x_R\|\} = \max\{\|(x_T + x_L)e^*e(x_T + x_L)^*\|^{1/2}, \|x_R^*ee^*x_R\|^{1/2}\} = \max\{\|x_Tv^*vx_T^* + x_Lw^*wx_L^*\|^{1/2}, \|x_R^*uu^*x_R\|^{1/2}\} = \max\{\|xtx^* + xlx^*\|^{1/2}, \|x^*rx\|^{1/2}\} = \max\{\|(t+l)x\|, \|rx\|\} = \|(t+l+r)x\| = \|x\|$, which shows that ι is an isometry. A similar calculation works at each matrix level, which concludes that ι is a complete isometry.