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Asymptotic-numerical approximations for highly oscillatory second-order differential equations by the phase function method

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ABSTRACT

Asymptotic approximations of "phase functions" for linear second-order differential equations, whose solutions are *highly* oscillatory, can be obtained using Borůvka's theory of linear differential transformations coupled to Liouville–Green (WKB) asymptotics. A numerical method, very effective in case of asymptotically polynomial coefficients, is extended to other cases of rapidly growing coefficients. Zeros of solutions can be computed without prior evaluation of the solutions themselves, but the method can also be applied to Initial- and Boundary-Value problems, as well as to the case of forced oscillations. Numerical examples are given to illustrate the performance of the algorithm. In all cases, the error turns out to be of the order of that made approximating the phase functions.

1. Introduction

Ordinary differential equations (ODEs) whose solutions are rapidly oscillatory are important in a number of applications, and their numerical treatment represents a challenging problem. Even in the simple case of second-order linear ODEs, such a task is highly nontrivial, and a number of methods have been proposed over the years to accomplish it. Among these, Lie group-based and Filon-type algorithms, e.g., seem to be very promising [7,8,24]. A completely different approach, to which we will refer to as the "phase function method", was proposed in [27,29] for the ODE

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$$y'' + q(x)y = 0 \tag{1}$$

on a half-line, when the coefficient, $q(x)$, belongs to one of the following classes of oscillatory equations:

- (1.a) $q(x) = a + \frac{b}{x} + g(x)$, where $a > 0$, $b \in \mathbf{R}$, $g(x)$ is suitably smooth and $g(x) = \mathcal{O}(x^{-p})$, $p \in \mathbf{R}$, $p > 1$, as $x \to +\infty$, [27];
- (1.b) $q(x) = c\, x^{\mu}[1 + g(x)]$, where $\mu, c \in \mathbf{R}^+$, $g(x)$ is suitably smooth and $g(x) = o(1)$ as $x \to +\infty$, [29].

These cases include, respectively, that of an *asymptotically constant* coefficient, and that of an *asymptotically polynomial* coefficient. In [27,29], convergence results were established and numerical examples provided. Note that the form of the ODE in (1) is general, since any (linear) term containing a first-order derivative can be removed either by a change of independent or dependent variable, see [3], e.g.

In this paper, we consider a third class of equations like (1), whose solutions are even more wildly oscillatory, namely the class

- (1.c) $q(x) = c\, e^{ax}[1 + g(x)]$, where $a > 0$, $c > 0$, $g(x)$ is suitably smooth, and $g(x) = o(1)$ as $x \to +\infty$ (but many other classes of rapidly growing coefficients could be handled).

Hereafter we set $c = 1$, since a different positive c amounts merely to change the x-scale. It should be observed that setting $x = \log t$ takes equation (1) with this coefficient into

$$t^2 \frac{d^2Y}{dt^2} + t\frac{dY}{dt} + t^a[1 + G(t)]\, Y = 0,$$

for $Y(t) := y(\log t)$, where $G(t) := g(\log t) = o(1)$ as $t \to +\infty$, and then, setting $Y(t) := t^{-1/2}Z(t)$, we obtain for $Z(t)$ the ODE

$$\frac{d^2Z}{dt^2} + Q(t)Z = 0, \quad Q(t) := t^{a-2} + \frac{1}{4t^2} + t^{a-2}G(t). \tag{2}$$

Note that the condition $\mu > 0$ in (1.b), requires $a > 2$, if one wants to apply the results valid for case (1.b) to case (1.c). When $a = 2$, we fall in case (1.a), but when $0 < a < 2$, we are not in such case. We will exploit Theorem 3.2 of [27,29] to infer what is established by our Theorem 3.1 below whenever $a > 2$, and resort to our Theorem 3.2 below otherwise.

Being $\liminf_{x\to\infty} x^2 q(x) > 1/4$, $\liminf_{t\to\infty} t^2 Q(t) > 1/4$, cases (1) and (2) are both oscillatory on the positive half-line [14, Theorem 7.1, p. 362] (or being $x^2 q(x) \ge (1 + \delta)/4$, for some $\delta > 0$). According to the Sturm comparison theorem [14, Ch. 11], all solutions of (1) with (1.b) or (1.c) exhibit a very rapidly oscillatory behavior. Consequently, the numerical treatment of such problems by traditional (time-stepping) methods represents a difficult task, and requires special discretization methods, see [10,17,25,18,19], e.g., and references therein. A number of alternatives have been proposed over the years [20,8]. Some methods as, e.g., Magnus, Cayley, and Neumann methods, are related to Lie groups techniques [20], others, like Filon and Filon-type methods, rest on suitable interpolations [7,8]. In an application to certain *forced* oscillator equations, relevant to electronic circuit simulations, Filon-type quadratures have been used quite successfully [7,8]. Similar problems have been treated efficiently by still another approach, based on modulated Fourier series [9]. In general, *asymptotics* and *symbolic* manipulations have proved to be very useful, and have been extensively exploited.

The inherent difficulties in handling rapidly oscillatory solutions with time-stepping methods strongly affects, in particular, the numerical approximation of *zeros* of solutions, at least when one tries to compute them through the prior evaluation of the solutions themselves. In [13], the authors proposed a fast marching