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Essential spectrum of a periodic waveguide with non-periodic perturbation ☆

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ABSTRACT

We consider the spectral Dirichlet–Laplacian problem on a domain which is formed from a periodic waveguide Π perturbed by non-compact, non-periodic changes of geometry. We show that the domain perturbation causes an addition to the essential spectrum, which consists of isolated points belonging to the discrete spectrum of a model problem. This model problem is posed on a domain, which is just a compact perturbation of Π. We discuss the position of the new spectral components in relation to the essential spectrum of the problem in Π.

1. Introduction and formulation of the problems

We consider the effect of non-compact domain perturbations to the essential spectrum of the Dirichlet–Laplacian. To describe our result, we assume that a 1-periodic quasicylinder Π in the Euclidean space $\mathbb{R}^d$, $d \geq 2$, as well as a sequence $(\ell_j)_{j=1}^{\infty}$ of natural numbers tending to $+\infty$ be given. The domain Π is perturbed by an infinite family of identical cells, such that the neighboring ones are situated at the distance ℓ_j of each other; see Fig. 1.1, 4.1. Our main result, Theorem 3.1, states that the essential spectrum, denoted later by $\sigma_{\mathrm{ess}}(A_\bullet)$, of the perturbed problem is the union of two components: the first one consists of the essential spectrum of the unperturbed problem and the second one of the discrete spectrum of a model problem, which is a spectral problem on the domain Π perturbed only by a single cell.

One of the conclusions is that the result does not depend on the growth rate of the sequence $(\ell_j)_{j=1}^{\infty}$; in particular, the domain perturbation is non-periodic and can be made as "sparse" as one wishes. However, it is

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essential for the result and its proof that the sequence $(\ell_j)_{j=1}^{\infty}$ is unbounded. The result should be compared with the papers [6], [1], [20], which contain analysis of the essential spectra of elliptic boundary problems in doubly periodic planar domains with domain perturbation consisting of semi-infinite open waveguides. These perturbations differ from those in the present paper, as they still have periodic structure with the same periodicity dimension as the intact domain and thus are rather analogous with the case of a constant sequence $(\ell_j)_{j=1}^{\infty}$. In [6] it shown that the essential spectrum consist of two components, one coming from the corresponding problem in the unperturbed domain and another one related to a family of model problems on the periodic domain perturbation. In both [6] and [20] there are explicit examples how the insertion of open waveguides into the domain increases the essential spectrum.

We proceed to detailed descriptions of the domains and equations to be investigated. The waveguide Π is assumed to be a domain in $\mathbb{R}^d$, $d \geq 2$, in particular a connected set, such that the boundary $\partial\Pi$ is a smooth $(d-1)$-dimensional surface. Moreover, Π is assumed to be a 1-periodic quasicylinder contained inside a circular cylinder of radius $R > 0$,

$$\Pi \subset \{x = (y, z) \in \mathbb{R}^{d-1} \times \mathbb{R} : |y| < R, z \in \mathbb{R}\}. \tag{1.1}$$

By the periodicity we mean that

$$\Pi = \{x = (y, z) \in \mathbb{R}^d : (y, z \pm 1) \in \Pi\},$$

and we denote the periodicity cell of Π by

$$\varpi = \{x \in \Pi : z \in (0, 1)\}. \tag{1.2}$$

and the translations of the cell $\varpi =: \varpi(1)$ by

$$\varpi(n) = \varpi + (0, \dots, 0, n-1) = \{x \in \Pi : z \in (n-1, n)\}, \quad n \in \mathbb{N}. \tag{1.3}$$

We consider the spectral Dirichlet problem for the Laplace operator $\Delta = \nabla \cdot \nabla$,

$$-\Delta u(x) = \lambda u(x) \text{ for } x \in \Pi, \quad u(x) = 0 \text{ for } x \in \partial\Pi. \tag{1.4}$$

The left-hand side of the variational formulation

$$(\nabla u, \nabla v)_\Pi = \lambda (u, v)_\Pi \quad \forall v \in H_0^1(\Pi) \tag{1.5}$$

contains a positive and closed bilinear form in $H_0^1(\Pi)$, and thus problem (1.4) is associated with a positive self-adjoint operator A with domain $\mathcal{D}(A) = H^2(\Pi) \cap H_0^1(\Pi)$; see [24, Thm. VIII.15], [4, Ch. 10]. Here, $(\cdot, \cdot)_\Pi$ is the natural scalar product in the Lebesgue space $L^2(\Pi)$, while $H^2(\Pi)$ and $H_0^1(\Pi)$ are standard Sobolev spaces, the latter with the homogeneous Dirichlet condition on $\partial\Pi$. It is known that the essential spectrum $\sigma_{\rm ess}(A)$ of A has the band-gap structure

$$\sigma_{\rm ess}(A) = \bigcup_{k \in \mathbb{N}} \beta(k), \tag{1.6}$$

where $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\beta(k)$ are compact intervals contained in $\mathbb{R}_+ = (0, \infty)$; see for example [25], [13]. The spectral bands $\beta(k)$ are described by means of a model spectral problem in the periodicity cell ϖ, see (1.9).

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