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Gaussian fluctuations for linear spectral statistics of Wigner beta ensembles

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A B S T R A C T

As an important topic in Mathematical Physics and statistics, random matrices theory has found uses in many aspects of modern physics and multivariate analysis. This paper is to investigate the Gaussian fluctuations for linear spectral statistics (LSS) of Wigner beta ensembles. We establish a central limit theorem (CLT) for LSS of Wigner quaternion matrices.

1. Introduction

Random matrices theory is known as an important topic in Mathematical Physics. It is shown to be inter-related with log-gases and the Calogero–Sutherland model. As an early introduced matrix models, the Gaussian β ensembles have been considered by a large number of authors. Here the special cases $\beta = 1, 2, 4$, known as Dyson's three-fold-way [9], correspond to Gaussian Orthogonal Ensemble (GOE), Gaussian Unitary Ensemble (GUE) and Gaussian Symplectic Ensemble (GSE) respectively. And the entries of a certain matrix in the above three ensembles are real, complex and quaternion standard gaussian variables. Since we can compute the explicit density functions of the joint distribution of eigenvalues, lots of properties of $G(O/U/S)E$ have been deduced by means of orthogonal polynomial. More details can be found in [14]. On the other hand, the central concept of the random matrix theory, as envisioned by E. Wigner, is the hypothesis that the distributions of eigenvalue spacings of large complicated quantum systems are universal in the sense that they depend only on the symmetry classes of the physical systems but not on detailed structures. This concept is also called "universality". By dropping the gaussian assumption, one consider the more general ensembles, the so called Wigner (β) ensembles where $\beta = 1, 2, 4$ (or equivalently,

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the Wigner real, complex, quaternion ensembles). For details in this direction, we refer the reader to [19,8,12,17] and references therein.

Also known as central limit theorems, global fluctuations for linear statistics have been of interest to the random matrix community for a long time. A variety of models and eigenvalue distributions have been studied from this point of view [10,1,18,6,16,13,11,2,3,5]. In this paper, as an extension of the results in [5], we will show the CLT for linear statistics of Wigner quaternion ensemble and thus fulfilling the corresponding CLT for Wigner (β) ensembles.

2. Some definitions and main results

We begin by a list of definitions and background that will be used in this paper. Set an ordered basis

$$\mathbb{e} = \mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbb{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbb{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where $i = \sqrt{-1}$ denotes the usual imaginary unit (here and in the rest of the paper, $\mathbf{I}_n$ denote the n dimensional identity matrix), then a quaternion can be represented as

$$q = a \cdot \mathbb{e} + b \cdot \mathbb{i} + c \cdot \mathbb{j} + d \cdot \mathbb{k} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix},$$

where a, b, c, d are real and $\alpha = a + bi$, $\beta = c + di$ are complex. The quaternion conjugate of q is defined as

$$q^Q = a \cdot \mathbb{e} - b \cdot \mathbb{i} - c \cdot \mathbb{j} - d \cdot \mathbb{k} = \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix} = q^*,$$

where $(\cdot)^*$ denote conjugate transform of a matrix.

We also write

$$qq^* = q^*q = \left(a^2 + b^2 + c^2 + d^2\right) \mathbf{I}_2 \triangleq \|q\|_Q^2 \mathbf{I}_2.$$

A Wigner quaternion matrix of size n is a quaternion self-dual Hermitian matrix where the upper-triangle entries are independent quaternion random variables. From [23], we know that a quaternion Hermitian matrix has $2n$ pairwise real eigenvalues. Suppose $\lambda_1^{\mathbf{Q}}, \lambda_1^{\mathbf{Q}}, \cdots, \lambda_n^{\mathbf{Q}}, \lambda_n^{\mathbf{Q}}$ are the $2n$ real eigenvalues of an $n \times n$ quaternion self-dual Hermitian matrix (a $2n \times 2n$ Hermitian matrix) $\mathbf{Q}$, then we call $\lambda_1^{\mathbf{Q}}\mathbf{I}_2, \lambda_2^{\mathbf{Q}}\mathbf{I}_2, \cdots, \lambda_n^{\mathbf{Q}}\mathbf{I}_2$ are the n quaternion eigenvalues of $\mathbf{Q}$.

In this paper, we define

$$\mathrm{E}(\|q\|_Q^k) = \mathrm{E}\left(\left(\sqrt{a^2 + b^2 + c^2 + d^2}\right)^k\right),$$

as the k-th norm moment of the quaternion random variable q.

For any function of bounded variation G on the real line, its Stieltjes transform is defined by

$$m_G(z) = \int \frac{1}{y - z} dG(y), \quad z \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} : \Im z > 0\}.$$

In [22], it is shown that under a Lindeberg type condition as $n \to \infty$, the Empirical Spectral Distribution (ESD) of a Wigner quaternion matrix whose entries being zero means and unit variances, convergence to the standard semicircular law F with density function