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Blow-up profiles and refined extensibility criteria in quasilinear Keller–Segel systems

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ABSTRACT

In this work we consider the system

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(S(u)\nabla v) & \text{in } \Omega\times(0,\infty) \\ v_t = \Delta v - v + u & \text{in } \Omega\times(0,\infty)\,, \end{cases}$$

for a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, where the functions D and S behave similarly to power functions. We prove the existence of classical solutions under Neumann boundary conditions and for smooth initial data. Moreover, we characterise the maximum existence time $T_{\max}$ of such a solution depending chiefly on the relation between the functions D and S: We show that a finite maximum existence time also results in unboundedness in L^p-spaces for smaller $p \in [1,\infty)$.

1. Introduction

The Keller–Segel systems considered in this work attempt to describe the behaviour of certain slime molds. In particular, given a position x and a time t, by $u(x,t)$ we denote the density of a cell population whose movement is motivated by the concentration $v(x,t)$ of a signal substance.

In these systems, which were proposed by Keller and Segel [17] in 1970 and of which there are several modifications (cf. e.g. Hillen and Painter [14]), the cross-diffusion makes solutions prone to blow-up and indeed blow-up detection is one of the most challenging tasks; to this day results remain fragmented. Even with the original system

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v) & \text{in } \Omega\times(0,\infty) \\ v_t = \Delta v - v + u & \text{in } \Omega\times(0,\infty) \end{cases}$$

there is no trivial answer on occurrences of blow-up, and if there is one, one often likes to know whether it arises in finite or infinite time. Beginning with a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary

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(and sufficiently regular initial data) we can state the following results: The case $n = 1$ has been studied (see [23]) with the result that there is no blow-up at all. For the two-dimensional setting we know that if the initial mass $\int_\Omega u_0$ is smaller than 4π, then solutions are bounded, for this we refer to [11] and [22], while for $n \geq 3$ a smallness condition on $\|u_0\|_{L^{\frac{n}{2}}(\Omega)} + \|v_0\|_{W^{1,n}(\Omega)}$ can be used to infer the existence of such a solution (see [4]). For larger initial data on the other hand we generally only know that there are blow-up solutions for which unboundedness can happen either in finite or infinite time [15].

In some cases the statements can be refined if we restrict ourselves to radially symmetric settings. For $\Omega = B_R(0) \subset \mathbb{R}^2$ and $\int_\Omega u_0 > 8\pi$ radially symmetric solutions that blow up in finite time have been found by [13] and [21] while in the case $\Omega = B_R(0) \subset \mathbb{R}^n$ and $n \geq 3$ even for small initial masses some solutions blow up in finite time (see [31]).

In this work we modify the first equation and for some bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, we consider the system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v) & \text{in } \Omega \times (0,\infty) \\ v_t = \Delta v - v + u & \text{in } \Omega \times (0,\infty) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(\cdot,0) = u_0,\ v(\cdot,0) = v_0 & \text{in } \Omega \end{cases} \tag{KS}$$

with nonnegative functions D and S. For a helpful overview of many models arising out of this fundamental description we also refer to the survey [1].

Several choices for these functions have been proposed and studied in recent years. One suggestion is to couple them via some function Q and the relations $D(u) = Q(u) - uQ'(u)$ and $S(u) = uQ'(u)$ for all $u \geq 0$. Here, Q is intended to describe the probability of a cell at (x,t) to find space nearby, [3] considers a decreasing function with decay at large densities as the best fit. In [32] an overview of hydrodynamic approaches or those involving cellular Potts models is given.

There are also authors who propose a signal dependence in D or S, that is to write e.g. $S(u,v)$ as done in [29], [14] and [25] to incorporate saturation effects or a threshold for the activation of cross-diffusion. For similar changes to D we refer to the works [9], [19], [27] and [26].

One set of choices has been of particular interest, namely where D and S behave like powers of u, and the result heavily depends on the relation of these two quantities. Setting

$$D(s) = (s+1)^{m-1} \text{ for all } s \in [0,\infty)$$

and

$$S(s) = s(s+1)^{\kappa-1} \text{ for all } s \in [0,\infty)$$

for some $m \in \mathbb{R}$ and $\kappa \in \mathbb{R}$ we find the following for $n \geq 2$: If $1 + \kappa - m < \frac{2}{n}$ and if the initial data are reasonably smooth, then we can find global classical solutions that are bounded [28] and this even remains true for general nonnegative functions D and S with

$$\frac{S(s)}{D(s)} \leq Cs^\alpha \text{ for all } s \geq 1$$

for some $C > 0$ and $\alpha < \frac{2}{n}$. On the other hand, if $1 + \kappa - m > \frac{2}{n}$ and if Ω is a ball, then for any $M > 0$ there are some $T \in (0,\infty]$ and a radially symmetric solution (u,v) in $\Omega \times (0,T)$ with $\int_\Omega u(\cdot,t) = M$ for all $t \in (0,T)$ such that u is not bounded in $\Omega \times (0,T)$ [30]. Once more there are also studies on more general choices of D and S (see [28], [30] as well as [18], [5], [24] and [16]) that find