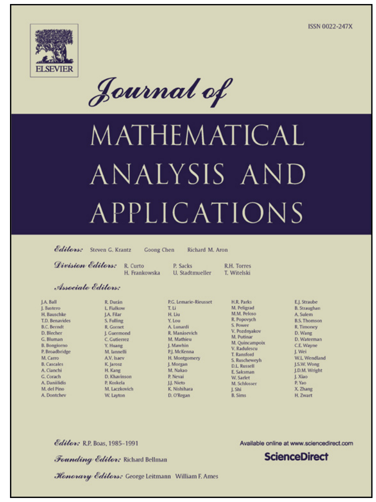

Boundedness of derivatives and anti-derivatives of holomorphic functions as a rare phenomenon

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Abstract

In this article we prove a general result which in particular suggests that, on a simply connected domain Ω in $\mathbb{C}$, all the derivatives and anti-derivatives of the generic holomorphic function are unbounded. A similar result holds for the operator $\widetilde{T}_N$ of partial sums of the Taylor expansion with center $\zeta \in \Omega$ at $z = 0$, seen as functions of the center ζ. We also discuss a universality result of these operators $\widetilde{T}_N$.

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1 Introduction

Let Ω be a domain in the complex plane and consider the space $\mathcal{H}ol(\Omega)$ of all the functions that are holomorphic on Ω with the topology of uniform convergence on compacta. In the first section of this article we show that, for a function $f \in \mathcal{H}ol(\Omega)$, the phenomenon of its k-th derivative or k-th anti-derivative being bounded on Ω is a rare phenomenon in the topological sense, provided that Ω is simply connected. We do this by using Baire's Theorem and we prove that the set $\mathcal{D}$ of all the functions $f \in \mathcal{H}ol(\Omega)$ with the property that, the derivatives and the anti-derivatives of f of all orders are unbounded on Ω, is a dense G_δ set in $\mathcal{H}ol(\Omega)$.

If a function f is holomorphic in an open set containing ζ, then $S_N(f,\zeta)(z)$ denotes the N-th partial sum of the Taylor expansion of f with center ζ evaluated at z. If Ω is a simply connected domain and $\zeta \in \Omega$, we define the class $U(\Omega,\zeta)$ as follows:

Definition 1.1. *Let $U(\Omega,\zeta)$ denote the set of all functions $f \in \mathcal{H}ol(\Omega)$ with the property that, for every compact set $K \subset \mathbb{C}$, $K \cap \Omega = \emptyset$, with K^{c} connected, and for every function h which is continuous on K and holomorphic in the interior of K, there exists a sequence $\{\lambda_n\} \in \{0,1,2,...\}$ such that*

$$\sup_{z \in K} |S_{\lambda_n}(f,\zeta)(z) - h(z)| \longrightarrow 0, \quad n \to \infty.$$