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Bohr phenomenon for subordinating families of certain univalent functions

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ABSTRACT

In this article we develop an elegant method to compute sharp Bohr radius for subordination classes of certain univalent functions.

1. Introduction

The Bohr phenomenon is named after Harald Bohr [16], who obtained the following remarkable result in 1914.

Theorem A. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in the open unit disk $\mathbb{D}$ and $|f(z)| < 1$ for all $z \in \mathbb{D}$, then*

$$\sum_{n=0}^{\infty} |a_n| r^n \leq 1 \tag{1.1}$$

for all $z \in \mathbb{D}$ with $|z| = r \leq 1/3$.

This constant $1/3$ is known as *Bohr radius* and in the above mentioned case this can not be improved. Bohr originally obtained this inequality for $r \leq 1/6$ which was sharpened to $r \leq 1/3$ by Wiener, Riesz and Schur independently. The problem was considered by Bohr while working on the absolute convergence problem for Dirichlet series of the form $\sum a_n n^{-s}$, but presently it has become an area of interest in itself. In a more general setting, Bohr inequality for mappings from $\mathbb{D}$ into domains other than $\mathbb{D}$ is widely researched upon in recent times, which is a motivation for the present article also. Apart from this, numerous other

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intriguing aspects of the study of Bohr phenomenon can be found in the articles [5,6,11,15,22,23]. Also, the recent survey article [4] and the references therein contain a detailed account of the advances regarding Bohr phenomenon.

We now present here the definition of subordination for analytic functions which is necessary for our further discussions in this article. For two analytic functions f and g in $\mathbb{D}$, we say g is subordinate to f if there exists a function ϕ, analytic in $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(z)| < 1$, satisfying $g = f \circ \phi$. Throughout this article we denote g is subordinate to f by $g \prec f$. Also the class of functions g subordinate to a fixed function f will be denoted by $S(f)$. Let the Taylor series expansions for f and g in a neighborhood of origin be

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.2}$$

and

$$g(z) = \sum_{k=0}^{\infty} b_k z^k \tag{1.3}$$

respectively. Now if f is normalized by $f(0) = f'(0) - 1 = 0$ then f has the following expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.4}$$

The concept of Bohr phenomenon for the class of functions which are defined by subordination was introduced in [1]. We say that $S(f)$ has Bohr phenomenon if for any $g \in S(f)$, where f and g have the Taylor expansions of the form (1.2) and (1.3) respectively in $\mathbb{D}$, there is a r_0, $0 < r_0 \le 1$ so that

$$\sum_{k=1}^{\infty} |b_k| r^k \le d(f(0), \partial\Omega)\,, \tag{1.5}$$

for $|z| = r < r_0$. Here $d(f(0), \partial\Omega)$ denotes the Euclidean distance between $f(0)$ and the boundary of domain Ω. Here and hereafter we will denote $\Omega := f(\mathbb{D})$, and the boundary of Ω will be denoted by $\partial\Omega$. It is clear from the definition that whenever $\Omega = \mathbb{D}$, $d(f(0), \partial\Omega) = 1 - |f(0)|$ and in this case (1.5) reduces to (1.1). In [1, Theorem 1] it was shown that $S(f)$ has Bohr phenomenon when f is univalent in $\mathbb{D}$. In particular the following theorem was established:

Theorem B. *If $g(z) = \sum_{k=0}^{\infty} b_k z^k \in S(f)$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is univalent in $\mathbb{D}$, then*

$$\sum_{k=1}^{\infty} |b_k| r^k \le d(f(0), \partial\Omega)\,, \tag{1.6}$$

for $|z| = r \le r_0 = 3 - 2\sqrt{2}$, and this r_0 is sharp for the Koebe function $k(z) = z/(1-z)^2$.

In addition it was remarked that if f is a convex univalent function, i.e. if $f(\mathbb{D})$ is a convex domain, then the radius r_0 in Theorem B can be improved to $1/3$ which is also sharp. It is seen that whenever a function g maps $\mathbb{D}$ into a domain Ω other than $\mathbb{D}$, then in a general sense the Bohr inequality (1.5) can be established if g can be recognized as a member of $S(f)$, f being the covering map from $\mathbb{D}$ onto Ω satisfying $f(0) = g(0)$. A number of results have been obtained using this idea (cf. [1–3,7]), and therefore the study of Bohr phenomenon for $S(f)$ where f belongs to a particular class of univalent functions can be