J. Math. Anal. Appl. 462 (2018) 1–25

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A new result for global existence and boundedness of solutions to a parabolic–parabolic Keller–Segel system with logistic source

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ARTICLE INFO

Article history:
Received 11 August 2017
Available online 3 February 2018
Submitted by P. Sacks

Keywords:
Chemotaxis
Global existence
Logistic source

ABSTRACT

We consider the following fully parabolic Keller–Segel system with logistic source

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + au - \mu u^2, & x\in\Omega, t>0, \\ v_t = \Delta v - v + u, & x\in\Omega, t>0, \end{cases} \tag{KS}$$

over a bounded domain $\Omega\subset\mathbb{R}^N (N\geq 1)$, with smooth boundary $\partial\Omega$, the parameters $a\in\mathbb{R}, \mu>0, \chi>0$. It is proved that if $\mu>0$, then (KS) admits a global weak solution, while if $\mu>\frac{(N-2)_+}{N}\chi C_{\frac{N}{2}+1}^{\frac{1}{\frac{N}{2}+1}}$, then (KS) possesses a global classical solution which is bounded, where $C_{\frac{N}{2}+1}^{\frac{1}{\frac{N}{2}+1}}$ is a positive constant which is corresponding to the maximal Sobolev regularity. Apart from this, we also show that if $a=0$ and $\mu>\frac{(N-2)_+}{N}\chi C_{\frac{N}{2}+1}^{\frac{1}{\frac{N}{2}+1}}$, then both $u(\cdot,t)$ and $v(\cdot,t)$ decay to zero with respect to the norm in $L^\infty(\Omega)$ as $t\to\infty$.

1. Introduction

The Keller–Segel model (see [18,19]) has been introduced in order to explain cell aggregation due to chemotaxis by means of a coupled system of two equations: a drift-diffusion type equation for the cell density u, and a reaction diffusion equation for the chemoattractant concentration v, that is, (u,v) satisfies

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v), & x\in\Omega, t>0, \\ v_t = \Delta v + u - v, & x\in\Omega, t>0. \end{cases} \tag{1.1}$$

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https://doi.org/10.1016/j.jmaa.2018.01.064
0022-247X/

The Keller–Segel models (1.1) and its variants have been extensively studied by many authors over the past few decades. The readers are referred to [1,13,14] more details descriptions of the models and their developments. The striking feature of Keller–Segel models is the possibility of blow-up of solutions in a finite (or infinite) time (see, e.g., [14,26,52]), which strongly depends on the space dimension. It is known that the model has only bounded solutions if $N = 1$ [28,57] (except in some extreme nonlinear degenerate diffusion model [5]); if $N = 2$, there exists a threshold value for the initial mass that decides whether the solutions can blow up or exist globally in time ([15,24,34]); when $N \geq 3$, there is no such threshold ([49,52]). The quasi-linear version together with the signal being consumed by the cells should be mentioned in the related works of this direction, and we refer to Cieślak et al. [5,6,8], Winkler et al. [1,36,48,56] and Zheng et al. [63,68] for more deep details.

In order to investigate the growth of the population, a considerable effort has been devoted to Keller–Segel models with the logistic term. For example, Winkler ([50]) proposed and investigated the following fully parabolic Keller–Segel system with logistic source

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + f(u), \quad x\in\Omega, t>0,\\ \tau v_t = \Delta v + u - v, \quad x\in\Omega, t>0,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0, \quad x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x), \quad \tau v(x,0) = \tau v_0(x), \quad x\in\Omega \end{cases} \tag{1.2}$$

with $\tau = 1$, where, $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial\nu}$ denotes the derivative with respect to the outward normal vector ν of $\partial\Omega$. The kinetic term f describes cell proliferation and death (simply referred to as growth). Hence, many efforts have been made first for the linear chemical production and the logistic source:

$$f(u) = au - \mu u^2. \tag{1.3}$$

During the past decade, the Keller–Segel models of type (1.2) have been studied extensively by many authors, where the main issue of the investigation is whether the solutions of the models are bounded or blow-up (see e.g., Cieślak et al. [9,5–7], Burger et al. [2], Calvez and Carrillo [3], Keller and Segel [18,19], Horstmann et al. [14–16], Osaki [28,27], Painter and Hillen [30], Perthame [31], Rascle and Ziti [33], Wang et al. [44,45], Winkler [47,48,50–52,54], Zheng [65]). If $\tau = 0$, (1.2) is referred as a simplified parabolic–elliptic chemotaxis system which is physically relevant when the chemicals diffuse much faster than the cells do. Tello and Winkler ([39]) mainly proved that the weak solutions of (1.2) ($\tau = 0$ in (1.2)) exist for arbitrary $\mu > 0$ and that they are smooth and globally bounded if the logistic damping effect satisfies $\mu > \frac{(N-2)_+}{N}\chi$. However, it was shown in some recent studies that the nonlinear diffusion (see Mu et al. [45,66]) and the (generalized) logistic damping (see Winkler [51], Li and Xiang [23], Zheng [59]) may prevent the blow-up of solutions.

Turning to the parabolic–parabolic system ($\tau = 1$ in (1.2)), for any $\mu > 0$, it is known, at least, that all solutions of (1.2) are bounded when $N = 1$ (see Osaki and Yagi [28]) or $N = 2$ (see Osaki et al. [27]). In light of deriving a bound for the quantity

$$\Sigma_{k=0}^{m} b_k \int_\Omega u^k |\nabla v|^{2m-2k}$$

with arbitrarily large $m \in \mathbb{N}$ and appropriately constructed positive $b_0, \ldots, b_m$, Winkler ([50]) proved that (1.2) admits a unique, smooth and bounded solution if μ is **large enough** and $N \geq 1$. However, he did not give the lower bound estimation for the logistic source. If $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a smooth and bounded **convex**