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Stability of ordered equilibria

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ABSTRACT

In this article we give an elementary method to investigate linear stability of equilibria of finite dimensional dynamical systems. In particular, under general hypotheses, the equilibria can be organised in an *ordered chain* along which the determinant of the associated Jacobian matrix has alternating sign. We develop the idea in two and three-dimensional cases, and then give a result for general n-dimensional systems. We also apply the technique to some particular, well known dynamical systems.

1. Introduction

The stability of the equilibria of a given dynamical system is a fundamental information on that system. Typically, a dynamical system models a natural phenomenon and depends on a number of parameters, and hence the determination of the regions in the parameter space in which a particular equilibrium is stable or unstable is a fundamental question. For example, in epidemiology, the function of parameters that discriminates the stability of the disease free equilibrium is the *basic reproduction number*. In this article we try to give a method to deduce information about the stability of an equilibrium only having information on the stability of some other equilibria. In particular we show that, under very general hypotheses, the equilibria of a dynamical system can be ordered to form a *chain of equilibria* $E_1 < E_2 < \cdots < E_m$. When such a chain of ordered equilibria is determined, we can show that whenever one of them, say $E_{\bar{i}}$, is stable, than the neighbouring equilibria $E_{\bar{i}-1}$, $E_{\bar{i}+1}$ must necessarily be unstable. This is a generalisation of the elementary fact that the equilibria of a 1-dimensional system $\dot{x} = f(x)$, $x \in \mathbb{R}$, must necessarily be of alternating type stable/unstable (if the given function f changes sign in a neighbourhood of a zero).

The construction relies on very elementary arguments but, as far as we know, this result is not explicitly described in the literature. Moreover, as we will show, this result makes it very simple to determine the stability of equilibria without analysing the positivity of some functions, e.g. those used in the Routh–Hurwitz

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criterion [6]. This result is very useful in particular when the dynamical system has high dimension or it involves many parameters.

We prove our results first in dimension two (Section 2) and three (Section 4). Then, we give the general n-dimensional result (Section 6). We apply the method to well known two (Section 3) and three dimensional dynamical systems (Section 4). We also give a 4-dimensional example at the end of Section 6. In Section 7 we give some concluding remarks, emphasising in particular the strengths and weaknesses of the proposed method.

2. The two-dimensional case

Let us consider a 2-dimensional ordinary differential system

$$\dot{\boldsymbol{x}} = \boldsymbol{X}(\boldsymbol{x}) \tag{1}$$

with $\boldsymbol{x} = (x, y)^T$ in Ω, a domain of $\mathbb{R}^2$, and $\boldsymbol{X} = (X, Y)^T$ a function defined in Ω with values in $\mathbb{R}^2$. The above system, in components, may be written

$$\begin{cases} \dot{x} = X(x, y) \\ \dot{y} = Y(x, y). \end{cases} \tag{2}$$

Assume that $X, Y : \Omega \to \mathbb{R}$ are differentiable functions, and let

$$\boldsymbol{\gamma} : (a, b) \subset \mathbb{R} \to \Omega \subset \mathbb{R}^2$$

be the parametrisation of a regular differentiable curve in Ω with $\boldsymbol{\gamma}'(s) \neq 0$ for every $s \in (a, b)$, $\boldsymbol{\gamma}(a) \neq \boldsymbol{\gamma}(b)$ and such that

$$X(\boldsymbol{\gamma}(s)) = 0, \qquad \nabla X(\boldsymbol{\gamma}(s)) \neq 0,$$

for every $s \in (a, b)$. These conditions imply that $\boldsymbol{\gamma}$ is a *regular parametrization* of a null-cline of X, i.e. is a curve in the set $\{(x, y) \,|\, X(x, y) = 0\}$.

Definition 1. A chain of ordered equilibria is an ordered list of points

$$E_1 = \boldsymbol{\gamma}(s_1) < \cdots < E_m = \boldsymbol{\gamma}(s_m)$$

where $s_1 < \cdots < s_m$ are all the $s \in (a, b)$ such that $Y(\boldsymbol{\gamma}(s_i)) = 0$.

Let $\boldsymbol{J}(\boldsymbol{x})$ be the Jacobian matrix of the vector field $\boldsymbol{X}$, and $J(\boldsymbol{x})$ its determinant (the Jacobian determinant). By hypothesis, the function $X(\boldsymbol{\gamma}(s))$ is identically zero, while the function

$$h(s) = Y(\boldsymbol{\gamma}(s)) \tag{3}$$

not only is not identically zero, but it plays an important role in the determination of the stability of the equilibria. The equilibria correspond in fact to the values $s_1, ..., s_m$ such that $h(s_i) = 0$, and the derivative of h at such points is related to the Jacobian determinant.

Lemma 1. *Let $\boldsymbol{X}$ be a vector field on the plane, $\boldsymbol{\gamma}(s)$ be a parametrization of a null-cline of the first component of $\boldsymbol{X}$, and let $h(s) = Y(\boldsymbol{\gamma}(s))$ be the composition of $\boldsymbol{\gamma}$ with the other component of $\boldsymbol{X}$. If the curve $\boldsymbol{\gamma}(s)$*

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