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Asymptotic expansion for solutions of the Navier–Stokes equations with non-potential body forces

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ABSTRACT

We study the long-time behavior of spatially periodic solutions of the Navier–Stokes equations in the three-dimensional space. The body force is assumed to possess an asymptotic expansion or, resp., finite asymptotic approximation, in Sobolev–Gevrey spaces, as time tends to infinity, in terms of polynomial and decaying exponential functions of time. We establish an asymptotic expansion, or resp., finite asymptotic approximation, of the same type for the Leray–Hopf weak solutions. This extends previous results that were obtained in the case of potential forces, to the non-potential force case, where the body force may have different levels of regularity and asymptotic approximation. This expansion or approximation, in fact, reveals precisely how the structure of the force influences the asymptotic behavior of the solutions.

1. Introduction

We study the Navier–Stokes equations (NSE) for a viscous, incompressible fluid in the three-dimensional space, $\mathbb{R}^3$. Let $\mathbf{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$ denote the space and time variables, respectively. Let the (kinematic) viscosity be denoted by $\nu > 0$, the velocity vector field by $\mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3$, the pressure by $p(\mathbf{x}, t) \in \mathbb{R}$, and the body force by $\mathbf{f}(\mathbf{x}, t) \in \mathbb{R}^3$. The NSE which describe the fluid's dynamics are given by

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} = -\nabla p + \mathbf{f} \quad \text{on } \mathbb{R}^3 \times (0, \infty), \\ &\text{div } \mathbf{u} = 0 \quad \text{on } \mathbb{R}^3 \times (0, \infty). \end{aligned} \tag{1.1}$$

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The initial condition is

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}^0(\mathbf{x}), \tag{1.2}$$

where $\mathbf{u}^0(\mathbf{x})$ is a given divergence-free vector field.

In this paper, we focus on the case when the force $\mathbf{f}(\mathbf{x},t)$ and solutions $\mathbf{u}(\mathbf{x},t)$, $p(\mathbf{x},t)$ are L-periodic for some $L>0$. Here, a function $\varphi(\mathbf{x})$ is L-periodic if

$$\varphi(\mathbf{x}+L\mathbf{e}_j)=\varphi(\mathbf{x}) \quad \text{for all} \quad \mathbf{x}\in\mathbb{R}^3,\ j=1,2,3,$$

where $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is the standard basis of $\mathbb{R}^3$.

Denote $\Omega=(-L/2,L/2)^3$, and let

$$\mathbf{U}(t)=\frac{1}{L^3}\int_\Omega \mathbf{u}(\mathbf{x},t)d\mathbf{x},\quad \mathbf{V}(t)=\int_0^t \mathbf{U}(\tau)d\tau,\quad \mathbf{F}(t)=\frac{1}{L^3}\int_\Omega \mathbf{f}(\mathbf{x},t)d\mathbf{x}.$$

Define

$$\mathbf{v}(\mathbf{x},t)=\mathbf{u}(\mathbf{x}+\mathbf{V}(t),t)-\mathbf{U}(t),\quad P(\mathbf{x},t)=p(\mathbf{x}+\mathbf{V}(t),t).$$

This is a variation of the Galilean transformation.

Integrating the first equation of (1.1) over Ω yields $\mathbf{U}'(t)=\mathbf{F}(t)$. Then one can verify that $\mathbf{v}(\mathbf{x},t)$ and $P(\mathbf{x},t)$ are L-periodic and satisfy the following NSE

$$\begin{aligned}&\mathbf{v}_t+(\mathbf{v}\cdot\nabla)\mathbf{v}-\nu\Delta\mathbf{v}=-\nabla P+\mathbf{g},\\ &\operatorname{div}\,\mathbf{v}=0,\end{aligned}$$

where $\mathbf{g}(\mathbf{x},t)=\mathbf{f}(\mathbf{x}+\mathbf{V}(t),t)-\mathbf{F}(t)$.

Note, by the spatial periodicity of $\mathbf{u}$, that

$$\int_\Omega \mathbf{v}(\mathbf{x},t)d\mathbf{x}=\int_\Omega \mathbf{u}(\mathbf{x}+\mathbf{V}(t),t)d\mathbf{x}-\int_\Omega \mathbf{u}(x,t)d\mathbf{x}=0.$$

Similarly, $\mathbf{g}(\mathbf{x},t)$ is L-periodic and

$$\int_\Omega \mathbf{g}(\mathbf{x},t)d\mathbf{x}=0.$$

Therefore, we can assume that $\mathbf{u}(\mathbf{x},t)$ and $\mathbf{f}(\mathbf{x},t)$ in (1.1), for all $t\geq 0$, have zero averages over the domain Ω. Here, a function $\varphi(\mathbf{x})$ is said to have zero average over Ω if

$$\int_\Omega \varphi(\mathbf{x})d\mathbf{x}=0. \tag{1.3}$$

By rescaling the spatial and time variables, we assume throughout, without loss of generality, that $L=2\pi$ and $\nu=1$.