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Global strong solutions for the incompressible nematic liquid crystal flows with density-dependent viscosity coefficient ☆

Yang Liu [a,b,*]

[a] Department of Mathematics, Nanjing University, Nanjing 210093, PR China
[b] School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, PR China

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ABSTRACT

This paper is concerned with an initial–boundary value problem of the incompressible nematic liquid crystal flows with density-dependent viscosity in a smooth bounded domain $\Omega \subset \mathbb{R}^3$. The global well-posedness of strong solutions with large oscillations is established in vacuum, provided $\|\nabla u_0\|_{L^2} + \|\Delta d_0\|_{L^2}$ is suitably small with arbitrary large initial density, which extended the local strong solution by Gao et al. [12] to be a global one.

1. Introduction

We consider the following hydrodynamic system modeling the flow of nematic liquid crystal materials

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ \rho u_t + \rho(u \cdot \nabla)u - \mathrm{div}(\mu(\rho)\nabla u) + \nabla P = -\lambda \mathrm{div}(\nabla d \odot \nabla d), \\ \mathrm{div} u = 0, \\ d_t + u \cdot \nabla d = \gamma(\Delta d + |\nabla d|^2 d), \end{cases} \tag{1.1}$$

in $\Omega \times (0, \infty)$, together with the initial and boundary conditions

$$(\rho, u, d)|_{t=0} = (\rho_0, u_0, d_0), \quad \text{with } |d_0| = 1, \quad \mathrm{div} u_0 = 0, \quad \text{in } \Omega, \tag{1.2}$$

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* Correspondence to: Department of Mathematics, Nanjing University, Nanjing 210093, PR China.
E-mail address: liuyang19850524@163.com.

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$$u(x,t)=0, \quad \frac{\partial d}{\partial \nu}(x,t)=0, \quad \text{on } \partial\Omega\times(0,\infty), \tag{1.3}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$ whose unit outward normal is ν. Here $u\in\mathbb{R}^3$ represents the velocity field of the flow, $d\in\mathcal{S}^2$, the unit sphere in $\mathbb{R}^3$, represents the macroscopic molecular orientation of the liquid crystal material, $\rho\in\mathbb{R}^+$ and $P\in\mathbb{R}$ are scalar functions, respectively, denoting the density of the fluid and the pressure arising from the usual assumption of incompressibility $\mathrm{div}u=0$. The positive constants λ and γ represent viscosity of fluid, competition between kinetic and potential energy, and microscopic elastic relaxation time respectively. The viscosity coefficient $\mu=\mu(\rho)$ is a general function of density, which is assumed to satisfy

$$\mu\in C^1[0,\infty) \quad \text{and} \quad 0<\underline{\mu}\le\mu\le\overline{\mu}<\infty \quad \text{on } [0,\infty), \tag{1.4}$$

for some positive constant $\underline{\mu}$, $\overline{\mu}$. Without loss of generality, both λ and γ are normalized to 1. The symbol $\nabla d\odot\nabla d$, which exhibits the property of the anisotropy of the material, denotes the $n\times n$ matrix whose (i,j)-th entry is given by $\partial_i d\cdot\partial_j d$, for $i,j=1,2,3$.

System (1.1)–(1.3) is a simplified version of the Ericksen–Leslie model, which reduces to the Ossen–Frank model in the static case, for the hydrodynamics of nematic liquid crystals developed by Ericksen [9] and Leslie [18] in the 1960's, but it still retains most important mathematical structures as well as most of the essential difficulties of the original Ericksen–Leslie model. Both the full Ericksen–Leslie model and the simplified version are the macroscopic continuum description of the time evolution of the materials, under the influence of both the flow velocity field u and the microscopic orientation configurations d of rod-like liquid crystals. Mathematically, system (1.1)–(1.3) is a strongly coupled system between the nonhomogeneous incompressible Navier–Stokes equations and the transported heat flows of harmonic map, and thus, its mathematical analysis is full of challenges.

When d is a constant vector and $|d|=1$, the system (1.1)–(1.3) reduces to the well-known nonhomogeneous incompressible Navier–Stokes equations. In the case that the viscosity μ is a constant and the initial density has a uniform positive lower bound, Kazhikov [1,17] established the global existence of weak solutions, and proved that there exists a unique local strong solution for arbitrary initial data with global existence of large strong solutions in $\mathbb{R}^2$. However in $\mathbb{R}^3$ the global well-posedness results were obtained only for small solutions. These results require relatively high regularity of the density, though. It is worthwhile to emphasize that for smooth densities with vacuum states, with the initial compatibility conditions

$$-\mu\Delta u_0+\nabla p_0=\sqrt{\rho_0}g \quad \text{and} \quad \mathrm{div}u_0=0 \quad \text{in } \Omega \tag{1.5}$$

for some $(p_0,g)\in H^1\times L^2$, Cho–Kim [3] proved the existence and uniqueness of local strong solutions in bounded domains or the whole space. Furthermore, global strong small solutions were obtained by Craig et al. [5]. Subsequently, without the initial compatibility conditions (1.5), Liang [23] proved the local strong solutions on the whole two-dimensional space with vacuum as far field density. Lv et al. [26] extended this result to global one and obtained some decay estimate of solutions. Recently, with the help of the Lagrangian formulation, Danchin–Mucha [6] get the local well-posedness with piecewise constant initial density ($\rho_0\in L^\infty(\mathbb{R}^3)$). Under additional assumption that the initial velocity is small and the density is close enough to a positive constant, they get the unique global solution. Similar results with lower regularity can be found in [13,27,2]. In case of density-dependent viscosity, Lions [24, Chapter 2] established the global existence of weak solutions to nonhomogeneous Navier–Stokes equations in any space dimensions for the initial density allowing vacuum. Cho–Kim [4] used the condition

$$-\mathrm{div}(2\mu(\rho_0)\nabla u_0)+\nabla p_0=\sqrt{\rho_0}g \quad \text{and} \quad \mathrm{div}u_0=0 \quad \text{in } \Omega \tag{1.6}$$