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The generalized hyperstability of general linear equations in quasi-Banach spaces

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ABSTRACT

In this paper, we study the hyperstability for the general linear equation in the setting of quasi-Banach spaces. We first extend the fixed point result of Brzdek et al. [5, Theorem 1] in metric spaces to b-metric spaces, in particular to quasi-Banach spaces. Then we use this result to generalize the main results on the hyperstability for the general linear equation in Banach spaces to quasi-Banach spaces. We also show that we can not omit the assumption of completeness in [5, Theorem 1]. As a consequence, we conclude that we need more explanations to replace a normed space by its completion in the proofs of some results in the literature.

1. Introduction

A definition of stability in the case of homomorphisms between groups was suggested by a problem posed by Ulam [19, page 64]. The first answer to Ulam's problem is the result of Hyers. Many authors then studied Hyers–Ulam stability of the following Cauchy equation.

$$f(x+y) = f(x) + f(y), \quad x, y \in X. \tag{1.1}$$

The distinguished results may be collected as in following Theorem 1.1, where the case $p = 0$ is due to Hyers [12, Theorems 1 & 2], the case $0 < p < 1$ is due to Aoki [2, Theorem on page 64], the case $p > 1$ is due to Gajda [10, Theorem 2], the case $p < 0$ is due to Lee [14, Theorem 5] and Brzdek [4, Theorem 1.2].

Theorem 1.1. *Let X, Y be two real normed spaces and $f : X \to Y$ be a function satisfying the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \alpha(\|x\|^p + \|y\|^p) \tag{1.2}$$

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for all $x, y \in X \setminus \{0\}$, where α and p are real constants with $\alpha > 0$ and $p \neq 1$. Then the following statements hold.

(1) *If $p \geq 0$ and Y is complete, then there exists a unique solution $T : X \to Y$ of (1.1) such that*

$$\|f(x) - T(x)\| \leq \frac{\alpha}{1 - 2^{p-1}} \|x\|^p \text{ for all } x \in X \setminus \{0\}. \tag{1.3}$$

(2) *If $p < 0$ then f is* additive, *that is, (1.1) holds for all $x, y \in X \setminus \{0\}$.*

In the recent paper [1], Aiemsomboon and Sintunavarat proved new generalized hyperstability results for the general linear equation of the form

$$g(ax + by) = Ag(x) + Bg(y) \tag{1.4}$$

where X and Y are Banach spaces over fields $\mathbb{F}$ and $\mathbb{K}$ respectively, $g : X \to Y$ is a function and $a, b \in \mathbb{F}$, $A, B \in \mathbb{K} \setminus \{0\}$. Note that Cauchy equation (1.1) is a special case of the general linear equation (1.4). Moreover the equation (1.4) also includes Jensen equation for $a = b = A = B = \frac{1}{2}$, and the linear equation for $a = A$, $b = B$. In 2015 Piszczek [18] studied the hyperstability for the general linear equation (1.4). The main results of [18] were generalized in [1] recently. The main results of [1] were proved by using the following fixed point result of Brzdek et al. [5], where Y^U is the set of all functions from $U \neq \emptyset$ to $Y \neq \emptyset$.

Theorem 1.2 *([5], Theorem 1). Suppose that*

(1) *U is a nonempty set, Y is a complete metric space, and $\mathcal{T} : Y^U \to Y^U$ is a given function.*
(2) *There exist $f_1, \ldots, f_k : U \to U$ and $L_1, \ldots, L_k : U \to \mathbb{R}_+$ such that for all $\xi, \mu \in Y^U$ and $x \in U$,*

$$d\big((\mathcal{T}\xi)(x), (\mathcal{T}\mu)(x)\big) \leq \sum_{i=1}^{k} L_i(x) d\Big(\xi\big(f_i(x)\big), \mu\big(f_i(x)\big)\Big). \tag{1.5}$$

(3) *There exist $\varepsilon : U \to \mathbb{R}_+$ and $\varphi : U \to Y$ such that for all $x \in U$,*

$$d\big((\mathcal{T}\varphi)(x), \varphi(x)\big) \leq \varepsilon(x)$$

(4) *For every $x \in U$,*

$$\varepsilon^*(x) = \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(x) < \infty$$

where $(\Lambda\delta)(x) = \sum_{i=1}^{k} L_i(x)\delta\big(f_i(x)\big)$ for all $\delta : U \to \mathbb{R}_+$ and $x \in U$.

Then for every $x \in U$, the limit

$$\lim_{n\to\infty} (\mathcal{T}^n \varphi)(x) = \psi(x)$$

exists and the function $\psi : U \to Y$ so defined is a unique fixed point of $\mathcal{T}$ satisfying

$$d\big(\varphi(x), \psi(x)\big) \leq \varepsilon^*(x)$$

for all $x \in U$.