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Inequalities for integrals of modified Bessel functions and expressions involving them

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ABSTRACT

Simple inequalities are established for some integrals involving the modified Bessel functions of the first and second kind. In most cases, we show that we obtain the best possible constant or that our bounds are tight in certain limits. We apply these inequalities to obtain uniform bounds for several expressions involving integrals of modified Bessel functions. Such expressions occur in Stein's method for variance-gamma approximation, and the results obtained in this paper allow for technical advances in the method. We also present some open problems that arise from this research.

1. Introduction

1.1. Motivation through Stein's method for variance-gamma approximation

Stein's method [35] is a powerful technique in probability theory for deriving bounds for distributional approximations with respect to a probability metric, with applications in areas as diverse as random graph theory [4], number theory [20] and random matrix theory [13]. The method is particularly well developed for normal approximation (see the books [10,29]), and there is active research into extensions to non-normal limits; see the survey [32].

Recently, Stein's method has been extended to variance-gamma (VG) approximation [12,14]. The VG distribution (also known as the generalized Laplace distribution [24]) is commonly used in financial mathematics [26], and has recently appeared in several papers in the probability literature as a limiting distribution [1–3]. This is in part due to the fact that the family of VG distributions is a rich one, with special or limiting cases that include, amongst others, the normal, gamma, Laplace, product of zero mean normals and difference of gammas [14,24]. It is therefore of interest to develop Stein's method for VG approximation to put

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some of the existing literature on Stein's method into a more general framework, and, more importantly, to extend it to new limit distributions.

At the heart of Stein's method for VG approximation is the function $f_h : \mathbb{R} \to \mathbb{R}$ defined by

$$f_h(x) = -\frac{\mathrm{e}^{-\beta x} K_\nu(|x|)}{|x|^\nu} \int_0^x \mathrm{e}^{\beta t} |t|^\nu I_\nu(|t|) h(t)\,\mathrm{d}t - \frac{\mathrm{e}^{-\beta x} I_\nu(|x|)}{|x|^\nu} \int_x^\infty \mathrm{e}^{\beta t} |t|^\nu K_\nu(|t|) h(t)\,\mathrm{d}t,$$

where $\nu > -\frac{1}{2}$, $-1 < \beta < 1$, and $h : \mathbb{R} \to \mathbb{R}$ is smooth and such that $\mu(h) = 0$, for μ the VG probability measure. A crucial part of the method is to obtain uniform bounds, in terms of the supremum norms of h and its derivatives, for $f_h(x)$ and its first four derivatives. In order to obtain these bounds, new inequalities were derived for integrals of modified Bessel functions [15,16], which were then used in the papers [14,11] to bound derivatives of all order.

To obtain distributional approximations in stronger probability metrics (such as the Kolmogorov and Wasserstein metrics), alternative bounds for f_h and its derivatives are required, which have a different dependence on the function h. This is the focus of [17,18], and to achieve such bounds, new inequalities are required for certain expressions involving integrals of modified Bessel functions. In this paper, we establish uniform bounds for some of these terms. In particular, we shall focus on bounding expressions of the type

$$\frac{\mathrm{e}^{-\beta x} K_{\nu+1}(x)}{x^\nu} \int_0^x \mathrm{e}^{\beta t} t^{\nu+1} I_\nu(t)\,\mathrm{d}t, \qquad \frac{\mathrm{e}^{-\beta x} I_{\nu+1}(x)}{x^\nu} \int_x^\infty \mathrm{e}^{\beta t} t^{\nu+1} K_\nu(t)\,\mathrm{d}t, \tag{1.1}$$

$$\frac{\mathrm{e}^{-\beta x} K_{\nu+1}(x)}{x^{\nu-1}} \int_0^x \mathrm{e}^{\beta t} t^{\nu} I_\nu(t)\,\mathrm{d}t, \qquad \frac{\mathrm{e}^{-\beta x} I_{\nu+1}(x)}{x^{\nu-1}} \int_x^\infty \mathrm{e}^{\beta t} t^{\nu} K_\nu(t)\,\mathrm{d}t. \tag{1.2}$$

In [17,18], these bounds are used in the development of a framework for deriving Kolmogorov and Wasserstein distance error bounds for VG approximation via Stein's method. The case $\beta = 0$ is dealt with in [17] and the case $\beta \neq 0$ will be dealt with in [18]. In [17], this framework is applied to obtain explicit bounds for VG approximation in a variety of settings, including quantitative six moment theorems for the VG approximation of double Wiener–Itô integrals (see [12] for related results); VG approximation for a special case of the D_2 statistic for alignment-free sequence comparison [9,25]; and Laplace approximation of a random sum of independent mean zero random variables (see [31] for related results). Further applications will be given in the companion paper [18].

1.2. Summary of the paper

The approach we shall take to bounding these expressions is to first bound the integrals in (1.1) and (1.2). Closed form expressions for these integrals, in terms of modified Bessel functions and the modified Struve function $\mathbf{L}_\nu(x)$, do in fact exist if $\beta = 0$. In this case, the integrals in (1.1) take a very simple form (see (A.57) and (A.58)). For $x > 0$ and $\nu > -\frac{1}{2}$, let $\mathscr{L}_\nu(x)$ denote $I_\nu(x)$, $\mathrm{e}^{\nu\pi i} K_\nu(x)$ or any linear combination of these functions, in which the coefficients are independent of ν and x. From formula 10.43.2 of [30],

$$\int x^\nu \mathscr{L}_\nu(x)\,\mathrm{d}x = \sqrt{\pi} 2^{\nu-1} \Gamma(\nu + \tfrac{1}{2}) x \big(\mathscr{L}_\nu(x) \mathbf{L}_{\nu-1}(x) - \mathscr{L}_{\nu-1}(x) \mathbf{L}_\nu(x) \big). \tag{1.3}$$

There are no closed form expressions in terms of modified Bessel and Struve functions in the literature for the integrals in (1.1) and (1.2) for the case $\beta \neq 0$. Moreover, even when $\beta = 0$ the expression on the right-hand side of (1.3) is a complicated expression involving the modified Struve function $\mathbf{L}_\nu(x)$. This provides the