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Stationarity of the crack-front for the Mumford–Shah problem in 3D

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ABSTRACT

In this paper we exhibit a family of stationary solutions of the Mumford–Shah functional in $\mathbb{R}^3$, arbitrary close to a crack-front. Unlike other examples, known in the literature, those are topologically non-minimizing in the sense of Bonnet [4]. We also give a local version in a finite cylinder and prove an energy estimate for minimizers. Numerical illustrations indicate the stationary solutions are unlikely minimizers and show how the dependence on axial variable impacts the geometry of the discontinuity set. A self-contained proof of the stationarity of the cracktip function for the Mumford–Shah problem in 2D is presented.

Contents

1. Introduction and main statements

The Mumford–Shah functional

$$J(u,K) := \int\limits_{\Omega\setminus K} |\nabla u|^2 \ + \alpha(u-g)^2 \, dx + \beta \mathcal{H}^{N-1}(K),$$

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where $g \in L^\infty(\Omega)$ and $u \in H^1(\Omega \setminus K)$, was introduced in the context of image processing by Mumford and Shah in [16]. The idea is to find, by minimizing the functional $J(u, K)$ among couples (u, K), a "piecewise smooth" approximation of the given image data g, together with the location of its edges, represented by the set K. Actually, the concept of competing bulk and surface energies is much older and goes back to Griffith (see [12]), whose theory of brittle fracture is based on the balance between gain in surface energy and strain energy release. This idea translated to the Mumford–Shah functional was exploited in the now classical variational model for crack propagation by Francfort and Marigo [10], relying on the theory of free discontinuity problems starting from the pioneering works of De Giorgi and co-authors [7,8]. We refer to [1,11,6,9,14] for overview references about the Mumford–Shah functional.

In this article we will take $\alpha = 0$, $\beta = 1$, and will consider local minimizers defined as follows. Let $\Omega \subset \mathbb{R}^N$ be open and let (u, K) be a couple such that $K \subset \Omega$ closed and $u \in H^1(B \setminus K)$, for all balls $B \subset \Omega$. We consider the local Mumford–Shah energy in the ball B given by

$$J(u, K, B) := \int_B |\nabla u|^2 \, dx + \mathcal{H}^{N-1}(K \cap B).$$

We are interested in couples (u, K) that are locally minimizing in Ω, i.e. such that

$$J(u, K, B) \leq J(u', K', B)$$

for any $B \subset \Omega$ and for any competitor (u', K') which satisfies $u = u'$ in $\Omega \setminus B$ and $K = K'$ in $\Omega \setminus B$. In that case we will simply say that (u, K) is a Mumford–Shah minimizer.

As minimizers are known to be equivalently defined on the SBV space (functions of special bounded variations, see [1]), it is not restrictive to assume K being an $(N-1)$-rectifiable set.

It is then classical that any such minimizer will satisfy the Euler–Lagrange equation associated to the Mumford–Shah functional, which is given by the equation (see [1, Theorem 7.35])

$$\int_\Omega |\nabla u|^2 \operatorname{div} \eta - 2\langle \nabla u, \nabla u \cdot \nabla \eta \rangle \, dx + \int_K \operatorname{div}^K \eta \;\; d\mathcal{H}^{N-1} = 0, \tag{1.1}$$

for all $\eta \in C_c^1(\Omega)^N$. We have denoted by $\operatorname{div}^K \eta$ the tangential divergence of η on K, which is well defined $\mathcal{H}^{N-1}$-a.e. on the rectifiable set K. A couple (u, K) satisfying the equation (1.1) for all $\eta \in C_c^1(\Omega)^N$ will be called *stationary* in Ω.

One of the most famous example of Mumford–Shah minimizer in $\mathbb{R}^2$ (i.e. here $\Omega = \mathbb{R}^2$) is the so-called cracktip function, which is known to be the only non-constant element in the list of global minimizers of [4]. This function plays also a fundamental role in fracture theory. Namely, we define

$$K_0 :=]-\infty, 0] \times \{0\} \subset \mathbb{R}^2$$

and

$$\varphi_0(r, \theta) := \sqrt{\frac{2r}{\pi}} \sin(\theta/2) \quad r > 0, \quad \theta \in]-\pi, \pi[.$$

It has been proved in a famous 250-pages paper by Bonnet and David [5] that the couple (φ_0, K_0) is a Mumford–Shah minimizer in $\mathbb{R}^2$.

The cracktip function has been widely studied and offers many challenging mathematical questions that are also of great interest regarding to brittle fracture theory and crack propagation (see for instance [14]).