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Complex symmetric operators and interpolation

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ABSTRACT

In this note we study the interpolation properties of complex symmetric operators on Hilbert spaces. We show that under certain conditions the complex symmetricity property is preserved under quadratic interpolation. We apply this result to the study of complex symmetric Toeplitz operators on weighted Hilbert spaces of analytic functions.

1. Introduction

In this note we study the interpolation properties of complex symmetric operators. Let $T\colon \mathcal{H} \to \mathcal{H}$ be a bounded linear operator on a separable complex Hilbert space $\mathcal{H}$. We say that T is *complex symmetric* if there exists in $\mathcal{H}$ an orthonormal basis in which T has a self-transpose matrix representation. There exists also an equivalent definition. Recall that a *conjugation* is a linear operator $C\colon \mathcal{H} \to \mathcal{H}$ such that $C^2 = I$ and C is isometric. An operator T is called C*-symmetric* if $T = CT^*C$ and *complex symmetric* if there exists a conjugation C with respect to which T is C-symmetric.

The notion of complex symmetric operators was introduced in 2006 in [9] and has since then become a dynamically developing subject of study. The general properties of complex symmetric operators have been described (see for instance [11]) and complex symmetric operators have been characterized for special classes of operators. We point to the growing number of papers concerning complex symmetric composition operators (see for instance [8]) and Toeplitz operators (see [13]). Note also that [16] recently examined complex symmetric operators between different Hilbert spaces.

Complex symmetric operators constitute a very general class of operators; for example, normal operators, Hankel and compressed Toeplitz operators are all complex symmetric (see [10] or [12]).

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Our aim here is to discuss the interpolation properties of complex symmetric operators. Let us briefly explain the concept. Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be Hilbert spaces and T be an operator such that $T\colon \mathcal{H}_0 \to \mathcal{H}_0$ and $T\colon \mathcal{H}_1 \to \mathcal{H}_1$ are bounded. Assume that $T\colon \mathcal{H}_0 \to \mathcal{H}_0$ (respectively, $T\colon \mathcal{H}_1 \to \mathcal{H}_1$) is C_0-symmetric (respectively, C_1-symmetric). Does it hold that $T\colon \mathcal{H}_\theta \to \mathcal{H}_\theta$ is C_θ-symmetric, where $\mathcal{H}_\theta$ denotes the geometric interpolation outcome of $\mathcal{H}_0$ and $\mathcal{H}_1$ for some θ? Note that for the narrower class of normal operators, the answer to the corresponding problem (that is, is T normal on $\mathcal{H}_\theta$?) is affirmative only in the case of regular and ordered couples of Hilbert spaces (see [15]).

Our main result is Theorem 1, which says that under certain conditions if an operator T on a couple of Hilbert spaces is C-symmetric on extreme spaces, then it is C-symmetric on intermediate spaces (with the same C). We also present examples showing that in general being a C-symmetric operator is not an interpolation property.

2. Interpolation of operators

To keep the paper self-contained, we begin by recalling some basic definitions and notation from the interpolation theory (see, e.g., [4,5,14,18]). Banach spaces X_0 and X_1 are called a *Banach couple* (we write $\vec{X} = (X_0, X_1)$) if there exists a Hausdorff topological vector space $\mathcal{X}$ such that $X_j \hookrightarrow \mathcal{X}$, $j = 0, 1$.

A Banach couple (X_0, X_1) is called *regular* if $X_j^\circ = X_j$, where X_j° denotes the closure of $X_0 \cap X_1$ in X_j, $j = 0, 1$. A couple (X_0, X_1) is *ordered* if $X_0 \subset X_1$. If $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ are Banach couples and $T\colon X_0 + X_1 \to Y_0 + Y_1$ is a linear map such that $T|_{X_j} \in L(X_j, Y_j)$, $j = 0, 1$, then we write $T\colon \vec{X} \to \vec{Y}$. The space $L(\vec{X}, \vec{Y})$ of all operators $T\colon \vec{X} \to \vec{Y}$ is a Banach space equipped with the norm

$$\|T\| := \max_{j=0,1} \|T|_{X_j}\|_{L(X_j,Y_j)}.$$

For simplicity, we write $L(\vec{X})$ instead of $L(\vec{X}, \vec{X})$.

We let $\Delta(\vec{X}) := X_0 \cap X_1$ and $\Sigma(\vec{X}) := X_0 + X_1$ equipped with the standard norms. A Banach space X is called an *intermediate* space with respect to a Banach couple $\vec{X} = (X_0, X_1)$ if $\Delta(\vec{X}) \subset X \subset \Sigma(\vec{X})$. Banach spaces X and Y are said to be *interpolation spaces* with respect to $\vec{X}$ and $\vec{Y}$ if X and Y are intermediate with respect to $\vec{X}$ and $\vec{Y}$, respectively, and if T maps X into Y for every $T \in L(\vec{X}, \vec{Y})$. If in addition there exists $C > 0$ and $\theta \in (0, 1)$ such that

$$\|T\colon X \to Y\| \leqslant C \,\|T\colon X_0 \to Y_0\|^{1-\theta} \,\|T\colon X_1 \to Y_1\|^{\theta}$$

for every $T \in L(\vec{X}, \vec{Y})$, then X and Y are said to be *of exponent* θ (and *exact of exponent* θ if $C = 1$). The complex interpolation space $[\vec{X}]_\theta$ and the real interpolation space $\vec{X}_{\theta,q}$ with $1 \leqslant q \leqslant \infty$ are exact of exponent θ (see [4]).

In this note we are considering interpolation between Hilbert spaces. Let $\vec{\mathcal{H}} = (\mathcal{H}_0, \mathcal{H}_1)$ be a regular couple of Hilbert spaces (for the non-regular case see [7]). By the Riesz representation theorem (see, e.g., [7, p. 253] and [2, p. 261] for more details), there exists a unique, densely defined, positive injective operator A in $\mathcal{H}_0$ such that

$$\langle \xi, \eta \rangle_1 = \langle A^{1/2}\xi, A^{1/2}\eta \rangle_0, \quad \xi, \eta \in \Delta(\vec{\mathcal{H}}),$$

where $\operatorname{Dom} A^{1/2} = \Delta(\vec{\mathcal{H}})$. The operator A is bounded if and only if $\mathcal{H}_0$ is contained in $\mathcal{H}_1$. We define a new inner product on $\Delta(\vec{\mathcal{H}})$ by

$$\langle \xi, \eta \rangle_\theta = \langle A^{\theta/2}\xi, A^{\theta/2}\eta \rangle_0, \quad \theta \in (0, 1).$$