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Kernels and ranks of Hankel operators on the Dirichlet spaces ☆

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ABSTRACT

We consider Hankel operators and little Hankel operators on the Dirichlet space and pluriharmonic Dirichlet space of the unit ball. We first describe kernels of Hankel operators with pluriharmonic symbol. Next, we prove that there are no non-trivial finite rank Hankel operators with pluriharmonic symbol. Finally, we characterize finite rank little Hankel operators with pluriharmonic symbol. Our results show that there are several differences between the results on the Dirichlet space and pluriharmonic Dirichlet space.

1. Introduction

Let B be the unit ball in the complex n-space $\mathbb{C}^n$ and V be the Lebesgue volume measure on $\mathbb{C}^n$ normalized so that $V(B) = 1$. The Sobolev space $\mathscr{S}$ is the completion of the space of all smooth functions f on B for which

$$\|f\| = \left\{ \left| \int_B f\, dV \right|^2 + \int_B \left(|\mathcal{R}f|^2 + |\widetilde{\mathcal{R}}f|^2 \right) dV \right\}^{1/2} < \infty$$

where

$$\mathcal{R}f(z) = \sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i}(z), \quad \widetilde{\mathcal{R}}f(z) = \sum_{i=1}^{n} \overline{z_i} \frac{\partial f}{\partial \overline{z_i}}(z)$$

for $z = (z_1, \cdots, z_n) \in B$. The Dirichlet space $\mathscr{D}$ is then a closed subspace of $\mathscr{S}$ consisting of all holomorphic functions in $\mathscr{S}$. Also we will consider the pluriharmonic Dirichlet space. Recall that a twice continuously

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differentiable function u on B is said to be pluriharmonic if the one-variable function $\lambda \mapsto u(a+\lambda b)$, defined for $\lambda \in \mathbb{C}$ such that $a + \lambda b \in B$, is harmonic for each $a \in B$ and $b \in \mathbb{C}^n$. The pluriharmonic Dirichlet space $\mathscr{D}_{ph}$ is a closed subspace of $\mathscr{S}$ consisting of all pluriharmonic functions in $\mathscr{S}$. Let P be the orthogonal projection from $\mathscr{S}$ onto $\mathscr{D}$. Also, we let Q be the orthogonal projection from $\mathscr{S}$ onto $\mathscr{D}_{ph}$. Put

$$\mathscr{L}^{1,\infty} = \left\{ \varphi \in \mathscr{S} \ : \varphi, \frac{\partial \varphi}{\partial z_j}, \frac{\partial \varphi}{\partial \bar{z}_j} \in L^\infty, j = 1, \cdots, n \right\}$$

where the derivatives are taken in the sense of distributions and $L^p = L^p(B, V)$ denotes the usual Lebesgue space on B.

For $u \in \mathscr{L}^{1,\infty}$, the *(big) Hankel operators* $\mathscr{H}_u : \mathscr{D} \to \mathscr{D}^\perp$ and $H_u : \mathscr{D}_{ph} \to \mathscr{D}_{ph}^\perp$ with symbol u are bounded operators defined respectively by

$$\mathscr{H}_u f = (I - P)(uf) \qquad \text{and} \qquad H_u \varphi = (I - Q)(\varphi u)$$

for $f \in \mathscr{D}$ and $\varphi \in \mathscr{D}_{ph}$. Also, the *little Hankel operators* $h_u : \mathscr{D} \to \mathscr{D}$ and $h_u^{ph} : \mathscr{D}_{ph} \to \mathscr{D}_{ph}$ with symbol u are bounded operators defined respectively by

$$h_u f = P(u\hat{f}) \qquad \text{and} \qquad h_u^{ph}\varphi = Q(u\hat{\varphi})$$

for functions $f \in \mathscr{D}$ and $\varphi \in \mathscr{D}_{ph}$. Here, for a function g on B, $\hat{g}$ is the function on B defined by $\hat{g}(z) = g(\bar{z})$ for $z \in B$. As we will see later, the little Hankel operator h_u^{ph} is just the Toeplitz operator with symbol u.

In this paper, we study kernels and ranks of the Hankel operators and little Hankel operators on the Dirichlet space and pluriharmonic Dirichlet space. The corresponding problems have been well studied on the Hardy space or the Bergman space. On the Hardy space of the unit disk, a well known Kronecker's theorem says that the little Hankel operator with symbol u has finite rank if and only if $p\hat{u}$ is analytic for some polynomial p; see [7] for example.

On the Bergman space of the unit disk, Das [3] proved that there are no finite rank Hankel operators with anti-holomorphic symbol and the kernel of a Hankel operator is trivial. Also, at the same paper, finite rank little Hankel operators and their kernels have been characterized. Later, Guo and Zheng [4] characterized finite rank little Hankel operators on the Bergman space of bounded symmetric domains.

Also, on the setting of the harmonic Bergman space, it has been shown that there are no nontrivial Hankel operators with harmonic symbol; see [9]. Recently, on the Dirichlet space of the unit disk, finite rank little Hankel operators have been characterized in [5] where an argument which can't be extended to the unit ball has been used.

In this paper, we study the corresponding problems for Hankel operators and little Hankel operators on the Dirichlet space and pluriharmonic Dirichlet space. In Section 3 we study kernels of Hankel operators. We first show that the kernel of a Hankel operator is trivial on the Dirichlet space $\mathscr{D}$; see Proposition 1. But, for the Hankel operator acting on the pluriharmonic Dirichlet space $\mathscr{D}_{ph}$, it has been shown that the kernel is not trivial in general by describing kernels of Hankel operators with pluriharmonic symbol; see Theorem 4.

In Section 4, we study the problem of when a Hankel operator has finite rank. We show that there are no nonzero finite rank Hankel operators with pluriharmonic symbol; see Theorem 9.

In Section 5, ranks of little Hankel operators have been studied. Using the known result [4] on the Bergman space mentioned above, we first characterize finite rank little Hankel operators with pluriharmonic symbol on the Dirichlet space; see Theorem 15. But, on the pluriharmonic Dirichlet space, we show that there are no nontrivial little Hankel operators with pluriharmonic symbol; see Corollary 17.

Our results mentioned above show that there are several differences and similarities between the results for the Hankel operators and little operators on the Dirichlet space and pluriharmonic Dirichlet space.