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Note

On densities of the product, quotient and power of independent subordinators

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A R T I C L E   I N F O

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A B S T R A C T

We obtain the closed form expressions for the densities of the product, quotient, power and scalar multiple of independent stable subordinators. Similar results for the independent inverse stable subordinators are discussed. This is achieved by expressing the densities of stable and inverse stable subordinators in terms of the Fox's H-function.

1. Introduction

A stable subordinator $D_\beta(t)$, with index β, is a one dimensional stable Lévy process with non-decreasing sample paths (see [1, pp. 49–52]). Subordinated processes have interesting probabilistic properties with applications in financial time series. The inverse stable subordinator is defined by $E_\beta(t) := \inf\{x > 0 : D_\beta(x) > t\}$. Some properties of the inverse stable subordinators and their applications are discussed in [11]. As statistical densities are basically elementary functions or the products of such functions, the theory of special functions is directly related to statistical distribution theory. The H-function, introduced by Fox [5], has numerous applications in the theory of statistical distributions and physical science. Some areas of astrophysics where the H-function appears naturally are the analytic solar model, pathway analysis, gravitational instability and reaction–diffusion problems (for more details on the H-function and its applications see [9]).

The main aim of this paper is to show that the random variable defined as the product, quotient, power or scalar multiple of independent stable subordinators has H-distribution. As the H-function has many known applications, we expect these results to have potential applications. For this purpose, we first obtain the closed form expression for the density of a β-stable subordinator in terms of the Fox's

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H-function for $\beta \in (0,1)$. Some known results such as the self-similarity property of a stable subordinator *etc.* are evident from such closed form. Similar representations for the densities of the tempered stable subordinator and the first-exit time of a stable subordinator, also known as the inverse stable subordinator, are obtained. Meerschaert and Scheffler [10] obtained the r-th moment of inverse stable subordinator as $\mathbb{E}\left(E_\beta(t)^r\right) = C(\beta, r)t^{\beta r}$, where $C(\beta, r)$ is some positive finite constant. Here, we explicitly determine this constant (see Remark 4.3) by using the Mellin transform of the H-function. The density $g_\beta(x,t)$ of $E_\beta(t)$ is connected to the density $f_\beta(x,1)$ of $D_\beta(1)$ by the following expression (see [10, Corollary 3.1], [11, Eq. (9)])

$$g_\beta(x,t) = \frac{t}{\beta}x^{-1-1/\beta} f_\beta\left(tx^{-1/\beta}, 1\right),$$

which can also be verified from the H-function representation of the densities of stable and inverse stable subordinators.

2. Preliminaries

We start with some definitions and identities, and set some notations required in the paper.

The Fox's H-function. This function is represented by the following Mellin–Barnes type contour integral (see [9, p. 2])

$$H(z) = \mathrm{H}^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_i, A_i)_{1,p}\\(b_j, B_j)_{1,q}\end{matrix}\right.\right] := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \chi(s) z^{-s}\,\mathrm{d}s, \tag{2.1}$$

where

$$\chi(s) = \frac{\prod_{i=1}^{n}\Gamma\left(1-a_i-A_i s\right)\prod_{j=1}^{m}\Gamma\left(b_j+B_j s\right)}{\prod_{i=n+1}^{p}\Gamma\left(a_i+A_i s\right)\prod_{j=m+1}^{q}\Gamma\left(1-b_j-B_j s\right)}.$$

An equivalent definition can be obtained on substituting $w = -s$ in (2.1). In the above definition, $i = \sqrt{-1}$, $z \neq 0$, and $z^{-s} = \exp\{-s(\ln|z| + i\arg z)\}$, where $\ln|z|$ represents the natural logarithm of $|z|$ and $\arg z$ is not necessarily the principal value. Also, m, n, p, q are integers satisfying $0 \le m \le q$ and $0 \le n \le p$ with $A_i, B_j > 0$ for $i = 1, 2, \dots, p$, $j = 1, 2, \dots, q$, and a_i's and b_j's are complex numbers. The path of integration, in the complex s-plane, runs from $c - i\infty$ to $c + i\infty$ for some real number c such that the singularity of $\Gamma\left(b_j + B_j s\right)$ for $j = 1, 2, \dots, m$ lie entirely to the left of the path and the singularity of $\Gamma\left(1 - a_i - A_i s\right)$ for $i = 1, 2, \dots, n$ lie entirely to the right of the path. An empty product is to be interpreted as unity.

The H-function distribution. This distribution was introduced by Carter and Springer [3] whose density is given by

$$f(x) = kH(\delta x) = \frac{\delta}{\chi(1)}\mathrm{H}^{m,n}_{p,q}\left[\delta x\left|\begin{matrix}(a_i, A_i)_{1,p}\\(b_j, B_j)_{1,q}\end{matrix}\right.\right], \quad x > 0,$$

where $\delta \neq 0$ and $k = \delta/\chi(1)$ is the normalizing constant. The cumulative distribution function of the H-function distribution is (see [2, p. 92], [4, p. 104])

$$F(x) = \frac{1}{\chi(1)}\mathrm{H}^{m,n+1}_{p+1,q+1}\left[\delta x\left|\begin{matrix}(1,1) & (a_i + A_i, A_i)_{1,p}\\(b_j + B_j, B_j)_{1,q} & (0,1)\end{matrix}\right.\right], \tag{2.2}$$

provided $-B_j^{-1}b_j < 1$ for all $j \in \{1, 2, \dots, m\}$. The Mellin and Laplace transforms of $H(\delta x)$ are given by (see [3], [9, pp. 47–50])