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On aperiodicity and hypercyclic weighted translation operators ☆

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ABSTRACT

We find several equivalent characterizations of aperiodicity of an element on a locally compact group G, and give an intuition for "how strong the aperiodicity of an element affects the existence of hypercyclic weighted translation operators." In fact, if a is an aperiodic element in G, then there exists a mixing, chaotic and frequently hypercyclic weighted translation $T_{a,w}$ on $L^p(G)$.

1. Introduction

This paper contains two parts. Firstly (Section 2), we characterize some equivalent statements of aperiodicity of an element on locally compact groups. Secondly (Section 3), we apply those equivalent statements of aperiodicity to give the existence of hypercyclic weighted translations. In some cases, we can actually find some explicit forms of hypercyclic weighted translations.

In the field of linear chaos, researchers focus on linear operators acting on a Banach space and discuss their dynamic properties like hypercyclicity and chaoticity. For our discussion, we focus on [3], (see also [1,2],) which characterizes the chaoticity and hypercyclicity of a weighted translation operator on the L^p space of a locally compact group.

An operator T on a Banach space X is called *hypercyclic* if there exists a vector $x \in X$ such that its orbit is dense in X (i.e. $\mathrm{orb}(T,x) := \{T^n x | n \in \mathbb{N}\}$ is dense in X). An operator T is called *weakly mixing* if $T \oplus T$ acting on $X \times X$ is hypercyclic. An operator T is called *mixing* if for any nonempty opens U, V in X, there exists $N \in \mathbb{N}$ such that $T^n U \cap V \neq \varnothing$ for all $n > N$. An operator T is called *chaotic* if it is hypercyclic and the set of periodic points is dense. According to [5, Proposition 9.3, p. 237], it is equivalent to define that an operator T is *frequently hypercyclic*, if there exists some $x \in X$ such that for any nonempty open

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subset U of X, $n_k = O(k)$, where n_k is a strictly increasing sequence of integers such that $T^{n_k}x$ is the k-th element lying in U.

The graph below displays some relations between these dynamical properties which have been discussed, see [5]:

$$\begin{tikzcd}
& \text{Frequently Hypercyclicity} \arrow[dr, Rightarrow] & & \\
\text{Frequent Hypercyclicity Criterion} \arrow[ur, Rightarrow] \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{Chaos} \arrow[r, Rightarrow] & \text{Weakly Mixing} \arrow[r, Rightarrow] & \text{Hypercyclicity} \\
& \text{Mixing} \arrow[ur, Rightarrow] & &
\end{tikzcd}$$

The graph above shows that the three conditions in the second column are implied by Frequent Hypercyclicity Criterion (in left). This follows from a useful theorem.

Theorem. *(Frequent Hypercyclicity Criterion, [5, Theorem 9.9 and Proposition 9.11]). Let T be an operator on a separable Fréchet space X. If there is a dense subset X_0 of X and a map $S : X_0 \to X_0$ such that, for any $x \in X_0$,*

(1) $\sum_{n=0}^{\infty} T^n x$ *converges unconditionally,*
(2) $\sum_{n=0}^{\infty} S^n x$ *converges unconditionally,*
(3) $TSx = x$,

then T is frequently hypercyclic. Moreover, T is also chaotic and mixing. In particular, it is also weakly mixing and hypercyclic.

In the paragraphs above, only general conclusions are described; in the following discussions, we will focus on the weighted translation operators.

Let G be a locally compact group and a be an element of G. The weighted translation operator $T_{a,w}$ is a bounded linear self-map on the Banach space $L^p(G)$ (by using the right Haar measure on G), for some $p \in [1, \infty)$, defined by

$$T_{a,w}(f)(x) := w(x)f(xa^{-1}),$$

where the weight w is a bounded continuous function from G to $(0, \infty)$. We denote $T_{a,1}$ by T_a so that $T_a f$ is the function f translated by a, while $T_{a,w}$ is a weighted translation operator.

To analyze $T_{a,w}$, we would like to classify some different topological properties of the elements in G. Let a be an element of G. We call it a *torsion* if it has finite order. An element a is *periodic* if the closed subgroup $G(a)$ generated by a (i.e. $G(a) = \overline{< a >}$) is compact in G. An element a is *aperiodic* if it is not periodic.

Throughout the paper, we assume that the locally compact group G is second countable and Hausdorff.

In [3, Lemma 2.1], we can find an equivalence statement of aperiodicity. Here we give another equivalence statement (Theorem 2.10), which is defined in Definition 2.3 and named as a *terminal pair*. To make this paper self-contained, we cite [3, Lemma 2.1] without proof.

Lemma. *[3, Lemma 2.1] An element a in a group G is aperiodic if, and only if, for each compact subset $K \subseteq G$, there exists $N \in \mathbb{N}$ such that $K \cap Ka^n = \varnothing$ for $n > N$.*