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Connected polynomials and continuity ☆

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As it is well-known, $f \in \mathbb{R}^{\mathbb{R}}$ is continuous if and only if f maps continua (compact, connected sets) to continua. The same holds for mappings between any two (real or complex) normed spaces. However, when we restrict ourselves to polynomials $P : E \to \mathbb{K}$, where E is a $\mathbb{K}$-normed space, then it was proved in 2012 that P is continuous if and only if it transforms compact sets into compact sets. Here we show that (if $\mathbb{K} = \mathbb{C}$) P is continuous if and only if it transforms connected sets into connected sets. Although we provide some partial results for $\mathbb{K} = \mathbb{R}$, the general case in the real setting remains still open.

1. Introduction

For convenience let us represent the space of all mappings $f : \mathbb{R} \to \mathbb{R}$ by $\mathbb{R}^{\mathbb{R}}$. The starting point of this paper is the following characterization of continuity in $\mathbb{R}^{\mathbb{R}}$:

Theorem 1.1. *A function $f \in \mathbb{R}^{\mathbb{R}}$ is continuous if and only if the following two conditions hold:*

(1) *For every compact set $C \subset \mathbb{R}$, we have that $f(C)$ is also compact, and*
(2) *for every connected set $C \subset \mathbb{R}$ (i.e., for every interval C), we have that $f(C)$ is also connected.*

In every basic course in Analysis of one real variable it is shown that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then it fulfills (1) and (2) above. For a proof of the reverse implication we refer to Velleman [12]. Actually, the following holds as a particular case of a result by Hamlett [9] (see also [10] and [13]):

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Theorem 1.2. *A functional $f : E \to \mathbb{R}$, where E is a normed space is continuous if and only if f transforms compact sets and connected sets of E into compact sets and connected sets of $\mathbb{R}$, respectively.*

In [8] the authors studied the algebraic size of the sets of mappings on the real line satisfying one and only one of the conditions (1) and (2). In order to understand the main conclusions of [8] it might be necessary to introduce the concept of lineability [3,2,1,4–6,8,7,11]:

Definition 1.3. If E is a linear space and λ is a cardinal number, we say that $M \subset E$ is λ-lineable if there exists a λ-dimensional linear subspace V of E such that $V \subset M \cup \{0\}$. If λ is an infinite cardinal, we simply say that M is lineable.

Recall that $\mathfrak{c}$, as usual, stands for the cardinality of $\mathbb{R}$. Then, in [8] the following was proved:

Theorem 1.4. *Let*

$$\mathcal{A}_1 = \{f \in \mathbb{R}^{\mathbb{R}} : \ f \text{ maps compact sets into compact sets}\} \quad \text{and}$$
$$\mathcal{A}_2 = \{f \in \mathbb{R}^{\mathbb{R}} : \ f \text{ maps connected sets into connected sets}\}.$$

Then both $\mathcal{A}_1 \setminus \mathcal{A}_2$ and $\mathcal{A}_2 \setminus \mathcal{A}_1$ are $2^{\mathfrak{c}}$-lineable, and this is optimal (in terms of dimension). Moreover, both $(\mathcal{A}_1 \setminus \mathcal{A}_2) \cup \{0\}$ and $(\mathcal{A}_2 \setminus \mathcal{A}_1) \cup \{0\}$ contain a $2^{\mathfrak{c}}$-dimensional space of nowhere continuous functions.

The above result is very much related to the question we consider in this paper. We need to introduce the notion of polynomial on a normed space. Given a normed space E over $\mathbb{K}$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, a map $P : E \to \mathbb{K}$ is an *n-homogeneous polynomial* if there is an n-linear mapping $L : E^n \to \mathbb{K}$ for which $P(x) = L(x, \ldots, x)$ for all $x \in E$. In this case it is convenient to write $P = \widehat{L}$. According to a well-known algebraic result, for every n-homogeneous polynomial $P : E \to \mathbb{K}$ there exists a unique symmetric n-linear mapping $L : E^n \to \mathbb{K}$ such that $P = \widehat{L}$. When this happens, L is called the polar of P.

We let $\mathcal{P}_a({}^n E)$, $\mathcal{L}_a({}^n E)$ and $\mathcal{L}_a^s({}^n E)$ denote respectively the linear spaces of all scalar-valued, n-homogeneous polynomials on E, the scalar-valued, n-linear mappings on E and the symmetric, scalar-valued, n-linear mappings on E. More generally, a map $P : E \to \mathbb{K}$ is a *polynomial of degree at most n* if

$$P = P_0 + P_1 + \cdots + P_n,$$

where $P_k \in \mathcal{P}_a({}^k E)$ $(1 \leq k \leq n)$, and $P_0 : E \to \mathbb{K}$ is a constant function. The polynomials of degree at most n on E are denoted by $\mathcal{P}_{n,a}(E)$.

Polynomials on a finite dimensional normed space are always continuous; however, the same statement is not valid for infinite dimensional normed spaces. Boundedness is a characteristic property of continuous polynomials on a normed space. In particular, $P \in \mathcal{P}_{n,a}(E)$ is continuous if and only if P is bounded on the open unit ball of E (denoted by B_E). This fact allows us to endow the space of contintinuous polynomials on E of degree at most n, represented by $\mathcal{P}_n(E)$, with the following norm:

$$\|P\| = \sup\{|P(x)| : x \in \mathsf{B}_E\}.$$

The space of continuous n-homogeneous polynomials on E is denoted by $\mathcal{P}({}^n E)$.

The following result refines Theorem 1.2 when restricting our attention to polynomials (see [8]):

Theorem 1.5. *If E is a real normed space, a polynomial $P \in \mathcal{P}_{n,a}(E)$ is continuous if and only if P transforms compact sets into compact sets.*