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On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind

Zhen-Hang Yang [a,b], Wei-Mao Qian [c], Yu-Ming Chu [a,*], Wen Zhang [d]

[a] *Department of Mathematics, Huzhou University, Huzhou 313000, China*
[b] *Customer Service Center, State Grid Zhejiang Electric Power Research Institute, Hangzhou 310009, China*
[c] *School of Distance Education, Huzhou Broadcast and TV University, Huzhou 313000, China*
[d] *Friedman Brain Institute, Icahn School of Medicine at Mount Sinai, New York, NY 10029, United States*

A R T I C L E I N F O

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A B S T R A C T

In the article, we prove that the double inequalities

$$\frac{1+(6p-7)r'}{p+(5p-6)r'}\frac{\pi\tanh^{-1}(r)}{2r} < \mathcal{K}(r) < \frac{1+(6q-7)r'}{q+(5q-6)r'}\frac{\pi\tanh^{-1}(r)}{2r},$$

$$\frac{qA(1,r)+(5q-6)G(1,r)}{A(1,r)+(6q-7)G(1,r)}L(1,r) < AGM(1,r) < \frac{pA(1,r)+(5p-6)G(1,r)}{A(1,r)+(6p-7)G(1,r)}L(1,r)$$

hold for all $r \in (0,1)$ if and only if $p \geq \pi/2 = 1.570796\cdots$ and $q \leq 89/69 = 1.289855\cdots$, where $\mathcal{K}(r) = \int_0^{\pi/2}(1-r^2\sin^2 t)^{-1/2}dt$ is the complete elliptic integral of the first kind, $\tanh^{-1}(r) = \log[(1+r)/(1-r)]/2$ is the inverse hyperbolic tangent function, $r' = \sqrt{1-r^2}$, and $A(1,r) = (1+r)/2$, $G(1,r) = \sqrt{r}$, $L(1,r) = (r-1)/\log r$ and $AGM(1,r)$ are the arithmetic, geometric, logarithmic and Gaussian arithmetic-geometric means of 1 and r, respectively.

1. Introduction

For $r \in (0,1)$, Legendre's complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ [13,14] of the first and second kinds are given by

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-r^2\sin^2(t)}},$$

* Corresponding author.
E-mail addresses: yzhkm@163.com (Z.-H. Yang), qwm661977@126.com (W.-M. Qian), chuyuming@zjhu.edu.cn (Y.-M. Chu), zhang.wen81@gmail.com (W. Zhang).

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$$\mathcal{E}(r)=\int_0^{\pi/2}\sqrt{1-r^2\sin^2(t)}dt,$$

respectively.

The Gaussian arithmetic-geometric mean $AGM(a,b)$ of two positive real numbers a and b is defined as the common limit of the sequences $\{a_n\}$ and $\{b_n\}$, which are given by

$$a_0=a,\quad b_0=b,\quad a_{n+1}=\frac{a_n+b_n}{2},\quad b_{n+1}=\sqrt{a_nb_n}.$$

The Gaussian and Landen identities [7] show that

$$AGM(1,r)=\frac{\pi}{2\mathcal{K}(r')},\quad \mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right)=(1+r)\mathcal{K}(r) \tag{1.1}$$

for all $r\in(0,1)$, where and in what follows $r'=\sqrt{1-r^2}$.

It is well known that $\mathcal{K}(r)$ and $\mathcal{E}(r)$ are the particular cases of the Gaussian hypergeometric function [6,29,33,36,44,45,47]

$$F(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}\quad(-1<x<1),$$

where $(a)_0=1$ for $a\neq 0$, $(a)_n=a(a+1)(a+2)\cdots(a+n-1)=\Gamma(a+n)/\Gamma(a)$ is the shifted factorial function and $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$ $(x>0)$ is the gamma function [25,26,52,53,55,58–60]. Indeed,

$$\mathcal{K}(r)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}r^{2n}, \tag{1.2}$$

$$\mathcal{E}(r)=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(n!)^2}r^{2n}.$$

The complete elliptic integrals and Gaussian hypergeometric function have many important applications in mathematics, physics and engineering. For example, the modulus of the plane Grötzsch ring can be expressed in terms of the complete elliptic integral of the first kind, and the complete elliptic integral of the second kind gives the formula of the perimeter of an ellipse. Moreover, Ramanujan modular equation and continued fraction in number theory are both related to the Gaussian hypergeometric function $F(a,b;c;x)$.

Recently, the bounds for the complete elliptic integrals have attracted the attention of many researchers. In particular, many remarkable inequalities and properties for $\mathcal{K}(r)$, $\mathcal{E}(r)$ and $F(a,b;c;x)$ can be found in the literature [3,4,8–12,16–24,27,28,32,35,37–43,46,48,50,51,54,56,57].

Carlson and Vuorinen [15], Vamanamurhty and Vuorinen [34], Qiu and Vamanamurthy [31] and Alzer [1] proved that the double inequalities

$$\frac{\log r'}{r'-1}<\mathcal{K}(r)<\frac{\pi\log r'}{2(r'-1)},$$

$$\left[1+\left(\frac{\pi}{4\log 2}-1\right)r'^2\right]\log\frac{4}{r'}<\mathcal{K}(r)<\left(1+\frac{1}{4}r'^2\right)\log\frac{4}{r'} \tag{1.3}$$

hold for all $r>0$.

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