J. Math. Anal. Appl. 462 (2018) 347–369

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Iterates of Markov operators and their limits

Johannes Nagler

Fakultät für Informatik und Mathematik, Universität Passau, Germany

ARTICLE INFO

Article history:
Received 9 August 2017
Available online 9 February 2018
Submitted by D. Khavinson

Keywords:
Iterates of Markov operator
Quasi-compactness
Browder operator
Spectral theory

ABSTRACT

It is well known that iterates of quasi-compact operators converge towards a spectral projection, whereas the explicit construction of the limiting operator is in general hard to obtain. Here, we show a simple method to explicitly construct this projection operator, provided that the fixed points of the operator and its adjoint are known which is often the case for operators used in approximation theory. We use an approach related to Riesz–Schauder and Fredholm theory to analyze the iterates of operators on general Banach spaces, while our main result remains applicable without specific knowledge on the underlying framework. Applications for Markov operators on the space of continuous functions $C(X)$ are provided, where X is a compact Hausdorff space.

The behaviour of the iterates of Markov operators has been studied extensively in modern ergodic theory, while in general the limiting operator is not explicitly given. A comprehensive overview on limit theorems for quasi-compact Markov operators can be found in Hennion and Hervé [8]. In this article, we construct the limit of the iterates of quasi-compact operators that satisfy a spectral condition. It will be shown under which conditions the limit exists and how the limiting projection operator can be explicitly constructed using the inverse of a Gram matrix. The explicit knowledge of the limiting operator is of interest in many applications.

This research is motivated by studying general Markov operators on the space of continuous functions $C(X)$, where X is a compact Hausdorff space. Lotz [16] has already shown uniform ergodic theorems for Markov operators on $C(X)$. For specific classes of operators, the limiting operator has been provided as shown for instance by Kelisky and Rivlin [12], Karlin and Ziegler [10] and Gavrea and Ivan [6,7]. Recently, Altomare [1] has shown a different approach using the concept of Choquet-boundaries and results from Korovkin-type approximation theory. Altomare et al. [2] have shown an application where they discussed differential operators associated with Markov operators, where also the knowledge of limit of the iterates is significant. Another application has been shown in the field of approximation theory, where the iterates can be used to prove lower estimates for Markov operators with sufficient smooth range, see Nagler et al. [18].

E-mail address: johannes.nagler@uni-passau.de.

https://doi.org/10.1016/j.jmaa.2018.02.012

It is worthwhile to mention that in most methods the limiting operator has to be known a priori. Here, we show an elegant extension to general Banach spaces for quasi-compact Markov operators. This extension provides a very general framework to explicitly construct the limiting operator with a simple method without prior knowledge of this operator.

After an introductory example, we introduce briefly our notation and recall the most important results that are necessary to prove our results. All of these results are well-known and can be found, e.g., in the classical books of Ruston [20], Rudin [19], Heuser [9]. In the next section, we discuss how the complemented subspace for some finite-dimensional eigenspace of an operator can be expressed in terms of the corresponding projection. We will start using the standard coordinate map to show the principle of our approach. Using a generalized version of the coordinate map we show conditions when the coordinate map on some eigenspace can be expressed in terms of a basis for this eigenspace and a basis of the corresponding eigenspace of the adjoint operator. These results are used to prove the limiting behaviour of the iterates of quasi-compact Markov operators.

1. An introductory example

We now demonstrate the simplicity of our result in a short example on $C([0,1])$, the space of continuous functions on the interval $[0,1]$. Note that in this case, also the method provided in [1, Section 3] can be used. Let n be a positive integer and suppose that $\{t_j\}_{j=1}^n$ form a partition of $[0,1]$, i.e. $0 = t_1 < t_2 < \cdots < t_n = 1$. We consider the positive finite-rank operator $T : C([0,1]) \to C([0,1])$, defined for $f \in C([0,1])$ by

$$Tf = \sum_{k=1}^{n} f(t_k) p_k, \tag{1}$$

where $p_1, \ldots, p_n \in C([0,1])$ are positive functions that form a partition of unity, i.e., $\sum_{k=1}^{n} p_k(t) = 1$ for all $t \in [0,1]$. It is easy to see that in this case $T1 = 1$ and $\|T\|_{op} = r(T) = 1$, where $r(T)$ is the spectral radius of T. Besides, we assume that

$$\sum_{k=1}^{n} t_k p_k(t) = t, \qquad t \in [0,1],$$

i.e., $Tf = f$ holds whenever f is a linear function. From that it follows already that $p_1(0) = p_n(1) = 1$ and T interpolates at 0 and 1, as

$$Tf(0) = \sum_{k=1}^{n} f(t_k) p_k(0) = f(t_1) = f(0), \quad Tf(1) = \sum_{k=1}^{n} f(t_k) p_k(1) = f(t_n) = f(1).$$

The introduced operator is a Markov operator, as it is a positive contraction and $T1 = 1$ holds. Two fixed points for T^* are given due to the interpolation at 0 and 1. If δ_0, δ_1 denote the continuous functionals that evaluate continuous functions at 0 and 1 respectively, then $\delta_0(Tf) = \delta_0(f)$ and $\delta_1(Tf) = \delta_1(f)$ holds for all $f \in C([0,1])$.

In the following, we want to answer the question whether the limit of the iterates T^m for $m \to \infty$ exists and if so to which operator the iterates converge. In Nagler [17] is has been shown that the partition of unity property, which is here equivalent to the ability to reproduce constant functions, guarantees that $\sigma(T) \subset B(0,1) \cup \{1\}$. To apply our main result, we have to specify the fixed point spaces of T and its adjoint T^*. Using the partition of unity property of T and the ability of T to reproduce linear functions as well as the ability to interpolate at the endpoints of the interval $[0,1]$, we derive the fixed-point spaces