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Fractional smoothness of images of logarithmically concave measures under polynomials ☆

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A B S T R A C T

We show that a measure on the real line, that is the image of a log-concave measure under a polynomial of degree d, possesses a density from the Nikolskii–Besov class of fractional order $1/d$. This result is used to prove an estimate for the total variation distance between such measures in terms of the Fortet–Mourier distance.

0. Introduction

Many fundamental problems of stochastic calculus involve investigation of the smoothness properties of measures of the form $\nu = \mu \circ f^{-1}$, i.e. measures induced by μ-measurable functions f with respect to a given measure μ on an infinite-dimensional space (e.g., the distribution of a stochastic process). In this paper we study the class of such measures ν induced by polynomials on spaces with logarithmically concave measures. Since all Gaussian measures are logarithmically concave, our results also applicable to Gaussian measures (e.g., to the Wiener measure). This class of distributions ν is of interest for many applications, because it contains typical statistics and because approximation by polynomials is a standard tool in many problems. Various properties of measures in the class under consideration have been studied in many works, see [13], [11], [14], [18], [28], [29], [32], [33] for the case of Gaussian measures and [1], [7], [8], [16], [26], [31] for the case of general logarithmically concave measures.

We recall that Nikolskii–Besov class $B^{\alpha}_{1,\infty}$, $\alpha \in (0, 1)$, consists of all functions $\psi \in L^1(\mathbb{R})$ for which there is number C_ψ such that

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$$\int_{\mathbb{R}} |\psi(x+h) - \psi(x)|dx \le C_\psi |h|^\alpha \quad \forall\, h \in \mathbb{R}.$$

Our first main result states that the density of a polynomial image of a log-concave measure always belongs to Nikolskii–Besov class $B^{1/d}_{1,\infty}$, where d is the degree of the polynomial. We also prove the following quantitative estimate (Corollary 4.2):

$$\sigma_f^{1/d} \int_{\mathbb{R}} |\rho_f(t+h) - \rho_f(t)|dt \le C(d)|h|^{1/d} \quad \forall h \in \mathbb{R},$$

where ρ_f is the density of the measure $\mu \circ f^{-1}$ for a log-concave measure μ and a polynomial f of degree d, and σ_f^2 is the variance of f. We note that even in the Gaussian case this result does not follow from [13]. This result is used to estimate the total variation distance between the distributions of polynomials in terms of the Fortet–Mourier distance (Corollary 4.4):

$$\|\mu \circ f^{-1} - \mu \circ g^{-1}\|_{\mathrm{TV}} \le C(d,a)\|\mu \circ f^{-1} - \mu \circ g^{-1}\|_{\mathrm{FM}}^{1/(1+d)},$$

provided that $\sigma_f, \sigma_g \ge a$. This estimate generalizes some recent results from [29], [31] and [13] to the case of log-concave measures. Moreover, even in the case of a Gaussian measure the power at the Fortet–Mourier distance in our estimate is better in comparison with the similar results from the cited papers.

The paper is organized in the following way. In Section 1 we give necessary definitions and some preliminary results needed in the proofs of the main results. The subsequent three sections contain the proofs of our results. An important tool in our approach is the so-called localization technique (Theorem 1.6) that enables to reduce certain high-dimensional inequalities to inequalities in low dimensions. This means that if we want to obtain a dimension-free estimate for the class of log-concave measures, we can prove a low-dimensional estimate and then use the localization technique to make it dimension-free. Let us outline some key steps in each section. The main tool of studying smoothness of induced distributions $\mu \circ f^{-1}$ is the classical Malliavin method [25] (see also [10]). The main idea of the method is to verify the estimates of the form

$$\int \varphi^{(n)}(f)d\mu \le C_n \sup_t |\varphi(t)|, \quad \forall \varphi \in C_0^\infty(\mathbb{R})$$

which yields the existence of the infinitely smooth density of the measure $\mu \circ f^{-1}$. However, in our case the density of a polynomial distribution may not even belong to the first Sobolev class, since it does not need to be even bounded (e.g., take the square of a standard normal random variable). So, in Section 2 we provide a similar sufficient Malliavin-type condition for the density of a measure on the real line to belong to the Nikolskii–Besov class (Lemma 2.1). We also deduce an estimate of the total variation distance in terms of the Fortet–Mourier distance for measures with densities from the Nikolskii–Besov class (Lemma 2.3). In Section 3 we apply localization technique to verify our Malliavin-type condition from Section 2 for the polynomial images of log-concave measures (Theorem 3.5). Note that one of the important assumptions of the classical Malliavin method is a certain nondegeneracy condition imposed on the mapping f that induces the distribution under consideration. In our case when f is a polynomial, we automatically have such a nondegeneracy condition in the form of the Carbery–Wright inequality (Theorem 1.4). Finally, in Section 4 we present our main results (Corollaries 4.2, 4.3, 4.4, and 4.5) for log-concave measures on infinite dimensional locally convex spaces that follow from the technical result of Theorem 3.5 and an approximation argument.

We would like to note that one could try to obtain similar results considering differential operator

$$L_\mu u := \Delta u + \nabla \log \rho_\mu \cdot \nabla u,$$