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Necessity of internal and boundary bulk balance law for existence of interfaces for an elliptic system with nonlinear boundary condition

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A B S T R A C T

For a system of stationary solutions to a reaction–diffusion equations with small diffusion coefficient and nonlinear flux boundary condition we prove that the bulk balance law, not only on the domain but on its boundary as well, is a necessary condition for formation of internal and boundary layers.

1. Introduction

A special type of parabolic partial differential equations, the so called reaction–diffusion equations, have long been used as mathematical models for many phenomena in physics, biology, chemistry and other fields alike.

Of special interest when studying such evolution systems in bounded euclidian domains supplied with no-flux boundary condition, are the non-constant stationary solutions. In some situations these spatially inhomogeneous solutions are characterized, for small values of a certain parameter, by inducing a partition in the domain where, except for a thin set – the so called transition layer region – the solutions are approximately constant. In a typical reaction of activator–inhibitor type this kind of solutions – often associated with pattern formation – would consist of regions, which may or may not intersect the boundary, on each of which the concentration of the activator is almost spatially uniform and the complementary regions where the inhibitor would have an almost uniform concentration. For definiteness we sometimes refer to these solutions as internal transition solutions.

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Loosing speaking, as this small parameter goes to zero, this one-parameter family of internal transition solutions approaches a piecewise constant function which partitions the domain into disjoint connected components. On each of these connected components, the limiting function assumes a constant value and the curves – or surfaces – separating these regions are called interfaces.

In the presence of nonlinear boundary flow, this situation may occur on the boundary of the domain as well, in which case the solutions are called boundary transition solutions. Interesting enough the limiting boundary interface may not be the trace of the limiting internal interface meaning that the two limiting interfaces, in the interior of the domain and on boundary, may be independent on each other (see [11], e.g., for this matter).

Of course it would be interesting to establish simple necessary conditions for existence of internal and boundary transition solutions for systems of this type. This is our task for the following elliptic system which determines the stationary solutions of the corresponding parabolic problem under non-linear flux boundary:

$$\begin{cases} \varepsilon\nabla\cdot(a(x)\nabla u)+f(x,u,\mathbf{v})=0, \text{ in } \Omega \\ \nabla\cdot(\mathbf{b}(x)\nabla \mathbf{v})+\mathbf{F}(x,u,\mathbf{v})=0, \text{ in } \Omega \\ \varepsilon a(x)\dfrac{\partial u}{\partial \nu}=g(x,u,\mathbf{v}), \qquad \text{on } \partial\Omega \\ \mathbf{b}(x)\dfrac{\partial \mathbf{v}}{\partial \nu}=\mathbf{G}(x,u,\mathbf{v}), \qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 2)$ is a bounded domain with C^2 boundary, ν the exterior normal vector field on $\partial\Omega$, f is a function in $C^1(\Omega\times\mathbb{R}\times\mathbb{R}^n)$, g is a function in $C^1(\partial\Omega\times\mathbb{R}\times\mathbb{R}^n)$. In addition, $\mathbf{F}=(f_1,\dots,f_n)$, $\mathbf{G}=(g_1,\dots,g_n)$, $\mathbf{b}=(b_1,\dots,b_n)$,

- $f_j\in C^1(\Omega\times\mathbb{R}\times\mathbb{R}^n)$, $j=1,\dots,n$,
- $g_j\in C^1(\partial\Omega\times\mathbb{R}\times\mathbb{R}^n)$, $j=1,\dots,n$,
- $a,b_j\in C^2(\overline{\Omega})$, $j=1,\dots,n$,
- $\nabla\cdot(\mathbf{b}(x)\nabla\mathbf{v})=(\nabla\cdot(b_1\nabla v_1),\dots,\nabla\cdot(b_n\nabla v_n))$,
- $\mathbf{b}(x)\dfrac{\partial\mathbf{v}}{\partial\nu}=\left(b_1\dfrac{\partial v_1}{\partial\nu},\dots,b_n\dfrac{\partial v_n}{\partial\nu}\right)$,
- $\exists\, M>0: M\leq a(x),\quad M\leq b_j(x)\ (j=1,\dots,n),\ \ \forall\, x\in\overline{\Omega}$.

Next we briefly describe our main results. Suppose that a family $\{(u_\varepsilon,\mathbf{v}_\varepsilon)\}$, $0<\varepsilon\leq\varepsilon_0$, of solutions to (1.1) develops internal and boundary transition layers, as $\varepsilon\to 0$, with interfaces $\mathcal{S}\subset\Omega$ and $\mathcal{C}\subset\partial\Omega$, respectively, and suppose further that $\partial\mathcal{S}=\mathcal{C}$. Roughly speaking, that is to say that there are continuous functions $\alpha,\beta:\Omega\longmapsto\mathbb{R}$ and $\widetilde{\alpha},\widetilde{\beta}:\partial\Omega\longmapsto\mathbb{R}$ such that, as $\varepsilon\to 0$, u_ε converges to α, in the L^1-topology, on one connected component of $\Omega\setminus\mathcal{S}$ and to β on the other, to $\widetilde{\alpha}$ on one connected component of $\partial\Omega\setminus\mathcal{C}$ and to $\widetilde{\beta}$ on the other. Moreover $\mathbf{v}_\varepsilon\longrightarrow\mathbf{v}_0$ uniformly in $\overline{\Omega}$, for some function $\mathbf{v}_0$ to be specified later.

Then, under these assumptions, we prove that necessarily

$$\int\limits_{\alpha(x)}^{\beta(x)} f(x,\xi,\mathbf{v}_0(x))d\xi=0,\quad \forall\, x\in\mathcal{S}\subset\Omega$$

and

$$\int\limits_{\widetilde{\alpha}(y)}^{\widetilde{\beta}(y)} g(y,\eta,\mathbf{v}_0(y))d\eta=0,\quad \forall\, y\in\mathcal{C}\subset\partial\Omega.$$